Spectral averaging techniques for Jacobi matrices

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Abstract

Spectral averaging techniques for one-dimensional discrete Schrödinger operators are revisited and extended. In particular, simultaneous averaging over several parameters is discussed. Special focus is put on proving lower bounds on the density of the averaged spectral measures. These Wegner type estimates are used to analyze stability properties for the spectral types of Jacobi matrices under local perturbations.

1 Introduction

Spectral averaging techniques for one-dimensional Sturm-Liouville or Jacobi operators have been developed and applied in various guises already for almost four decades (see *e.g.* [14] and references therein). The basic idea is that the Hamiltonian may depend on some parameters (such as boundary conditions, coupling constants and alike) and that the spectral measures averaged over these parameters are absolutely continuous w.r.t. the Lebesgue measure. The main object of this work is to give various criteria on local perturbations that, on top of that, guarantee that the Lebesgue measure is absolutely continuous w.r.t. to the averaged spectral measures, thus showing that they are equivalent. Applications of this equivalence concern spectral analysis. In fact, local perturbations may change drastically spectral properties of Jacobi operators, in particular, properties which are related to the singular part of the spectrum. Nevertheless, if one knows that some of these properties hold for sets of parameters which have a large measure, then it is possible to prove results about stability of them.

In Section 2 we include the necessary background for the spectral averaging techniques. The case of one-parameter spectral averaging discussed in Section 3 is a discrete version of results for Sturm-Liouville operators obtained in [2] (see also [3]), reformulated using a Birman-Schwinger operator instead of an associated regular problem. The results on averaging over several parameter in Section 4 are related to results of Wegner [16] on the density of states for (also multi-dimensional) random Schrödinger operators which have recently been made rigorous by Hislop and Müller [11]. However, we do not only deal with homogeneous operators, but allow that the randomness is only in a finite volume. In

terms of the strength of the disorder we estimate the size of this volume needed in order to insure that the averaged spectral measure is equivalent to the Lebesgue measure. We argue heuristically what an optimal estimate on this volume would be (which we were unable to prove) and explain how it would allow to correct the wrong weak-disorder scaling behavior of the bounds obtained in [16, 11]. Let us also cite [1] for further results on several parameter spectral averaging. Section 5 exhibits some application to the spectral analysis of the Jacobi operators studied in Section 4.

2 Recollection of basic formulas

This section is a review of several well-known results, which will be used below. Let $(t_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be sequences of respectively positive and real numbers, and $\alpha, \beta \in \mathbb{R}$. For a given $N \in \mathbb{N}$, the finite Jacobi matrix $H^N_{\alpha,\beta}$ with left boundary condition $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and right boundary condition $\beta \in (0, \pi)$ is an operator on the finite-dimensional Hilbert space $\ell^2(\{1, ..., N\})$ given by

$$(H_{\alpha,\beta}\phi)_n \;=\; t_{n+1}\phi_{n+1} + v_n\phi_n + t_n\phi_{n-1}\,, \qquad n = 1, \ldots, N\,, \tag{1}$$

where $t_1 = t_{N+1} = 1$, together with the boundary conditions

$$\sin(\alpha)\phi_1 - \cos(\alpha)\phi_0 = 0\,, \qquad \sin(\beta)\phi_{N+1} + \cos(\beta)\phi_N = 0\,.$$

The matrix written out explicitely is

$$H^N_{\alpha,\beta} \;=\; \begin{pmatrix} v_1 + \tan(\alpha) & t_2 & & & & \\ t_2 & v_2 & t_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & t_{N-1} & v_{N-1} & t_N & \\ & & & t_N & v_N + \cot(\beta) \end{pmatrix}. \tag{2}$$

Dirichlet boundary conditions are given if $\alpha = 0$ and $\beta = \frac{\pi}{2}$. We will also consider the limit $N \to \infty$ of semi-infinite Jacobi matrices. If $H_{\alpha,\beta}$ is in the Weyl limit point case at infinity, then there is a unique self-adjoint limit operator denoted by H_α, independent of β.

2.1 Transfer matrices

The transfer matrices are defined for any complex energy z as

$$\mathcal{T}^z_n \;=\; \begin{pmatrix} (z - v_n)\, t_n^{-1} & -t_n \\ t_n^{-1} & \mathbf{0} \end{pmatrix}\,, \qquad n = 1, \ldots, N\,. \tag{3}$$

Then we introduce the transfer matrices over several sites by

$$\mathcal{T}^z(n, m) \;=\; \mathcal{T}^z_n \cdot \ldots \cdot \mathcal{T}^z_{m+1}\,, \qquad n > m\,,$$

and $\mathcal{T}^z(n,n) = \mathbf{1}$. They allow to write out all those solutions of the finite difference equation $H^N_{\alpha,\beta}\phi^z(\alpha) = z\phi^z(\alpha)$ satisfying the left boundary condition:

$$\begin{pmatrix} t_{n+1}\phi^z_{n+1}(\alpha) \\ \phi^z_n(\alpha) \end{pmatrix} \;=\; \mathcal{T}^z(n,0) \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} . \tag{4}$$

The right boundary condition is satisfied precisely at the eigenvalues of $H^N_{\alpha,\beta}$. Hence $z \in \mathbb{R}$ is an eigenvalue of $H^N_{\alpha,\beta}$ if and only if for some $\lambda \neq 0$

$$\mathcal{T}^z(N,0) \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} \;=\; \lambda \begin{pmatrix} \cos(\beta) \\ \sin(\beta) \end{pmatrix} .$$

Let us introduce the following notations for the entries of the transfer matrix

$$\mathcal{T}^z(N,0) \;=\; \begin{pmatrix} a^z_N & b^z_N \\ c^z_N & d^z_N \end{pmatrix} . \tag{5}$$

As first formula, let us recall the result of a Wronskian calculation.

Proposition 1 *If $\phi^z_n = \phi^z_n(0)$ is the solution with initial conditions $\phi^z_1 = 1$ and $\phi^z_0 = 0$, then*

$$a^z_N\overline{c^z_N} - \overline{a^z_N}c^z_N \;=\; (z-\overline{z}) \sum_{n=1}^{N} |\phi^z_n|^2 .$$

Proof. It follows from (4) and the definition (5) that

$$a^z_N\overline{c^z_N} - \overline{a^z_N}c^z_N \;=\; t_{N+1}\phi^z_{N+1}\overline{\phi^z_N} - t_{N+1}\overline{\phi^z_{N+1}}\phi^z_N .$$

Replacing twice the Schrödinger equation $t_{N+1}\phi^z_{N+1} = (z-v_N)\phi^z_N - t_N\phi^z_{N-1}$ gives

$$a^z_N\overline{c^z_N} - \overline{a^z_N}c^z_N \;=\; (z-\overline{z})|\phi^z_N|^2 + t_N\phi^z_N\overline{\phi^z_{N-1}} - t_N\overline{\phi^z_N}\phi^z_{N-1} .$$

Iteration over N proves the formula.

□

2.2 Finite volume Green's function identities

The Green's function of $H^N_{\alpha,\beta}$ is defined by

$$G^N_{\alpha,\beta}(z,n,m) \;=\; \langle n|(H^N_{\alpha,\beta} - z)^{-1}|m\rangle ,$$

where $n,m = 1,\ldots,N$ and $\Im m(z) > 0$. For Dirichlet boundary conditions, we drop the indices α and β. Furthermore, we set $G^N_{\alpha,\beta}(z) = G^N_{\alpha,\beta}(z,1,1)$. The latter is linked to the spectral measure $\rho^N_{\alpha,\beta}$ of $H^N_{\alpha,\beta}$ w.r.t. the state $|1\rangle$ by

$$G^N_{\alpha,\beta}(z) \;=\; \int \rho^N_{\alpha,\beta}(dE)\, \frac{1}{E-z} .$$

Some connections of the Green's function to the transfer matrix are given in the following proposition (other relations can also be obtained, but will not be used here).

Proposition 2 *One has*

$$\frac{1}{a_N^z} \;=\; -\,G^N(z,1,N)\;, \qquad \frac{b_N^z}{a_N^z} \;=\; G^N(z,1,1)\;, \qquad \frac{c_N^z}{a_N^z} \;=\; -\,G^N(z,N,N)\;.$$

Proof. By Cramer's rule

$$G(z,1,N) \;=\; (-1)^{N+1}\,\frac{\det(H^N-z)_{1,N}}{\det(H^N-z)}\;,$$

where $\det(H^N-z)_{1,N}$ is the subdeterminant with the first row and the Nth column erased. Evaluation gives $\det(H^N-z)_{1,N} = t_2\cdots t_N$. Furthermore, developing by the last column, one gets

$$\det(H^N-z) = (v_N-z)\det(H^{N-1}-z) - t_N^2\det(H^{N-2}-z).$$

Hence $N\mapsto \det(H^N-z)$ satisfies the same recurrence relation as $N\mapsto (-1)^N t_2\cdots t_{N+1}\phi_{N+1}^z$ as can be deduced from the Schrödinger equation. Moreover, the initial conditions coincide, namely $\det(H^1-z) = v_1-z = -t_2\phi_2$ and $\det(H^2-z) = (v_1-z)(v_2-z)-t_2^2 = (-1)^2 t_2t_3\phi_3$. Therefore one deduces that $\det(H^N-z) = (-1)^N t_1\cdots t_{N+1}\phi_{N+1}^z = (-1)^N t_1\cdots t_N a_N^z$ and

$$G(z,1,N) \;=\; (-1)^{N+1}\,\frac{t_2\cdots t_N}{(-1)^N t_2\cdots t_N a_N^z} \;=\; -\,\frac{1}{a_N^z}.$$

For the second equality, let us start from

$$G(z,1,1) \;=\; \frac{\det(\tilde H^N-z)}{\det(H^N-z)}\;,$$

where $\tilde H^N$ is the $(N-1)\times(N-1)$ matrix obtained from H^N by restriction to the sites $2,\dots,N$. From the above calculation, we have that $(-1)^{N-1}t_2\cdots t_N\tilde a_N^z = \det(\tilde H^N - z)$ where $\tilde a_N^z$ is the upper left entry of the transfer matrix $\mathcal{T}_N^z\cdots\mathcal{T}_2^z$. By multiplication with $\mathcal{T}_1^z$ one readily verifies that $\tilde a_N^z = -b_N^z$. Thus replacing $\det(H^N-z) = (-1)^N t_1\cdots t_N a_N^z$ gives

$$G(z,1,1) \;=\; \frac{b_N^z}{a_N^z}\;.$$

Finally, again by Cramer's rule,

$$G(z,N,N) \;=\; \frac{\det(H^{N-1}-z)}{\det(H^N-z)}\;.$$

Using again twice the identity $\det(H^N-z) = (-1)^N t_1\cdots t_{N+1}\phi_{N+1}^z$ allows to conclude the proof of the last identity. □

Proposition 3 *The dependence of the Green's function $G_{\alpha,\beta}^N(z)$ on the boundary conditions is given by*

$$G_{\alpha,\beta}^N(z) \;=\; \frac{b_N^z - d_N^z\cot(\beta)}{a_N^z + b_N^z\tan(\alpha) - c_N^z\cot(\beta) - d_N^z\tan(\alpha)\cot(\beta)}\;.$$

Proof. The boundary conditions α, β can be incorporated in the potential values v_1, v_N as in (2), which is then understood to have Dirichlet boundary conditions. The resulting transfer matrix from 1 to N can be expressed in terms of the transfer matrix $\mathcal{T}^z(N,0)$:

$$\begin{pmatrix} 1 & -\cot(\beta) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_N^z & b_N^z \\ c_N^z & d_N^z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tan(\alpha) & 1 \end{pmatrix} . \tag{6}$$

Evaluating and extracting the upper left and right entries concludes the proof together with the second formula of Proposition 2. □

Proposition 4 *For $\Im m(z) > 0$,*

$$\int_0^\pi \frac{d\beta}{\pi}\, G^N_{\alpha,\beta}(z) \;=\; \frac{b_N^z + \imath d_N^z}{(a_N^z + b_N^z \tan(\alpha)) + \imath(c_N^z + d_N^z \tan(\alpha))} ,$$

and for $E \in \mathbb{R}$

$$\lim_{\epsilon \to 0} \Im m \int_0^\pi \frac{d\beta}{\pi}\, G^N_{\alpha,\beta}(E + \imath\epsilon) \;=\; \frac{1}{|a_N^E + b_N^E \tan(\alpha)|^2 + |c_N^E + d_N^E \tan(\alpha)|^2} . \tag{7}$$

Proof. Using (6) it is easy to deduce the formulas for arbitrary boundary α from the case $\alpha = 0$. Hence it is sufficient to consider the latter case. From Proposition 3 and a change of variables it follows that

$$\int_0^\pi \frac{d\beta}{\pi}\, G^N_{0,\beta}(z) \;=\; \int_0^\pi \frac{d\beta}{\pi}\, \frac{b_N^z \tan(\beta) - d_N^z}{a_N^z \tan(\beta) - c_N^z} \;=\; \int_{-\infty}^\infty \frac{dx}{\pi(1+x^2)}\, \frac{b_N^z x - d_N^z}{a_N^z x - c_N^z} .$$

The latter integral can be evaluated by a contour integral. The poles of the integrand are at $x = \imath, -\imath, \frac{c_N^z}{a_N^z}$. As $\frac{c_N^z}{a_N^z} = -G^N(z,N,N)$ is in the lower half-plane, the only pole in the upper half-plane is $x = \imath$. Hence the residue theorem directly implies the first formula of the proposition. The second one follows directly by calculating the imaginary part and using the fact that the coefficients $a_N^E, b_N^E, c_N^E, d_N^E$ are real and satisfy $a_N^E d_N^E - b_N^E c_N^E = 1$.

It follows immediately from (7) and the de la Vallée-Poussin theorem that $\int_0^\pi \frac{d\beta}{\pi}\, \rho^N_{\alpha,\beta}$ is absolutely continuous with density given by the r.h.s. of (7). □

2.3 Prüfer variables

In this section, the energy is real and hence we set $z = E \in \mathbb{R}$. For any fixed left boundary condition α, we define as [9] the Prüfer phases $\theta_n^E(\alpha)$ and Prüfer radius $R_n^E(\alpha)$ by

$$R_n^E \begin{pmatrix} \cos(\theta_n^E) \\ \sin(\theta_n^E) \end{pmatrix} \;=\; \mathcal{T}^E(n,0) \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} , \tag{8}$$

together with the condition $-\frac{\pi}{2} < \theta_{n+1}^E(\alpha) - \theta_n^E(\alpha) < \frac{3\pi}{2}$ and $\theta_0^E(\alpha) = \alpha$. Next we derive a few formulas used in the sequel.

Proposition 5

$$\int_0^\pi \frac{d\alpha}{\pi} \, \frac{1}{R_N^E(\alpha)^2} \;=\; 1 \, .$$

Proof. Setting $\mathcal{T} = \mathcal{T}^E(N,0)$ and $e_\alpha = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}$, the integral is given by

$$\int_0^\pi \frac{d\alpha}{\pi} \, \frac{1}{R_N^E(\alpha)^2} \;=\; \int_0^\pi \frac{d\alpha}{\pi} \, \frac{1}{\langle e_\alpha | \mathcal{T}^* \mathcal{T} | e_\alpha \rangle} \, .$$

Now $\mathcal{T}^*\mathcal{T}$ is a positive matrix of determinant 1, hence its eigenvalues are $\kappa, \frac{1}{\kappa} > 0$ and it is diagonalized by an orthogonal matrix. As $d\alpha$ is rotation invariant, it follows that

$$\int_0^\pi \frac{d\alpha}{\pi} \, \frac{1}{R_N^E(\alpha)^2} \;=\; \int_0^\pi \frac{d\alpha}{\pi} \, \frac{1}{\kappa \cos^2(\alpha) + \frac{1}{\kappa} \sin^2(\alpha)} \;=\; \int_{-\infty}^\infty \frac{dx}{\pi} \, \frac{1}{\kappa + \frac{1}{\kappa} x^2} \;=\; 1 \, ,$$

which concludes the proof. □

The first of the following two formulas was already proven in [9].

Proposition 6 (i) *One has*

$$R_N^E(\alpha)^2 \, \partial_E \theta_N^E(\alpha) \;=\; -\sum_{n=1}^N |\phi_n^E(\alpha)|^2 \, .$$

(ii) *For the derivative w.r.t. the potential value v_n with $n \leq N$, one has*

$$R_N^E(\alpha)^2 \, \partial_{v_n} \theta_N^E(\alpha) \;=\; |\phi_n^E(\alpha)|^2 \, .$$

Proof. Due to the special form (3) of the transfer matrices, item (i) follows directly from item (ii). Hence we focus on (ii). Furthermore, let us suppress the α in all notations. Deriving $\tan(\theta_N^E) = \frac{\phi_N^E}{t_{N+1}\phi_{N+1}^E}$ w.r.t. to v_n gives

$$\partial_{v_n} \theta_N^E(\alpha) \;=\; \frac{1}{1 + \left(\frac{\phi_N^E}{t_{N+1}\phi_{N+1}^E}\right)^2} \, \partial_{v_n} \frac{\phi_N^E}{t_{N+1}\phi_{N+1}^E} \, .$$

Evaluation and using $(R_N^E)^2 = (\phi_N^E)^2 + (t_{N+1}\phi_{N+1}^E)^2$ gives

$$\partial_{v_n} \theta_N^E(\alpha) \;=\; \frac{1}{(R_N^E)^2} \left[(\partial_{v_n}\phi_N^E)(t_{N+1}\phi_{N+1}^E) - (\phi_N^E)(\partial_{v_n} t_{N+1}\phi_{N+1}^E) \right] \, .$$

Now, as long as $n < N$, the term in the brackets can be evaluated by replacing twice the Schrödinger equation $t_{N+1}\phi_{N+1}^E = (E - v_n)\phi_N^E - t_N \phi_{N-1}^E$. At the first step one obtains

$$\partial_{v_n} \theta_N^E(\alpha) \;=\; \frac{1}{(R_N^E)^2} \left[(\partial_{v_n}\phi_{N-1}^E)(t_N\phi_N^E) - (\phi_{N-1}^E)(\partial_{v_n} t_N \phi_N^E) \right] \, ,$$

and iteration gives

$$\partial_{v_n}\theta_N^E(\alpha) \;=\; \frac{1}{(R_N^E)^2}\left[(\partial_{v_n}\phi_n^E)(t_{n+1}\phi_{n+1}^E) - (\phi_n^E)(\partial_{v_n}t_{n+1}\phi_{n+1}^E)\right] \;.$$

As $\partial_{v_n}\phi_n^E = 0$ and $\partial_{v_n}t_{n+1}\phi_{n+1}^E = -\phi_n^E$ by the Schrödinger equation, one can conclude the proof. □

Finally let us prove Carmona's formula [5] (which was rediscovered by Pearson [13] and proven by Simon in the discrete case [15]).

Proposition 7 *For any $E_0 < E_1$,*

$$\frac{1}{2}\left[\rho_\alpha([E_0,E_1]) + \rho_\alpha((E_0,E_1))\right] \;=\; \lim_{N\to\infty}\int_{E_0}^{E_1}\frac{dE}{\pi}\,\frac{\cos^2(\alpha)}{R_N^E(\alpha)^2}\;.$$

Proof. Integrating (7) w.r.t. energy, one gets

$$\frac{\pi}{2}\int_0^\pi\frac{d\beta}{\pi}\,\left[\rho_{\alpha,\beta}^N([E_0,E_1]) + \rho_{\alpha,\beta}^N((E_0,E_1))\right] \;=\; \int_{E_0}^{E_1} dE\,\frac{\cos^2(\alpha)}{R_N^E(\alpha)^2}\;,$$

where $\rho_{\alpha,\beta}^N$ is the spectral measure of $H_{\alpha,\beta}^N$ w.r.t. $|1\rangle$. Now let us take the limit $N\to\infty$ of this equation. On the l.h.s. one may invoke the dominated convergence theorem in order to take the limit under the integral. As $\rho_{\alpha,\beta}^N$ converges weakly to ρ_α for all β, the result follows immediately. □

3 One-parameter spectral averaging

Let us first recall the well-known result on averages over the left boundary condition [5, 14]. One considers $\alpha\in[0,\pi)\mapsto H_\alpha$ with corresponding spectral measures ρ_α. Then the spectral averaged measure $\rho = \int_0^\pi d\alpha\,\frac{1}{\cos^2(\alpha)}\,\rho_\alpha$ is equal to the Lebesgue measure (up to a factor π). In fact, Propositions 5 and 7 immediately allow to deduce this statement. As shows the form (2) of the Hamiltonian, this result can be understood as an average over the potential value at site 1 w.r.t. a particular density. As shown in the last remark of this section, averages over another single potential value (say at site N) can also be analyzed. We shall be interested in studying spectral averages in situations where several entries in the Jacobi matrix are varied. In this section, this is done with only one parameter, then in the next one with several parameters.

In order to single out the main mechanism, let us start with a more abstract statement on spectral averaging. Let $I=[\mu_0,\mu_1]$ be a finite interval of parameters and suppose given a differentiable family $\mu\in I\mapsto H(\mu)$ of semi-infinite Jacobi matrices with Dirichlet boundary conditions $\alpha=0$ and such that only the first N potential values $v_1,\ldots,v_N$ and off-diagonal terms $t_2,\ldots,t_N$ depend on μ. The associated spectral measure is denoted by ρ_μ, and the Prüfer phases by $\theta_N^E(\mu)$. Furthermore define the averaged spectral measure ρ by

$$\rho \;=\; \int_{\mu_0}^{\mu_1} d\mu\;\rho_\mu\;. \tag{9}$$

Proposition 8 *Suppose that for all $E \in [E_0, E_1]$:*
(i) *$\mu \in I \mapsto \theta_N^E(\mu)$ is strictly monotonous with bounded derivative,*
(ii) *$|\theta_N^E(\mu_1) - \theta_N^E(\mu_0)| > \pi$.*

Then, in the interval $[E_0, E_1]$, the averaged spectral measure ρ is equivalent to the Lebesgue measure.

Proof. Using Proposition 7, the dominated convergence theorem and Fubini's theorem, one gets

$$\rho([E_0, E_1]) \;=\; \lim_{L\to\infty} \int_{E_0}^{E_1} dE \int_{\mu_0}^{\mu_1} d\mu \; \frac{1}{(R_L^E(\mu))^2} \;.$$

Next let us set for $L > N$:

$$R_{L,N}^E(\theta) \;=\; \left\| \mathcal{T}^E(L,N) \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \right\| \;.$$

This does not depend on μ, and

$$R_L^E(\mu) \;=\; R_{L,N}^E(\theta_N^E(\mu))\; R_N^E(\mu) \;. \tag{10}$$

Now the continuity of $\mu \mapsto H(\mu)$ gives the bounds

$$0 \;<\; C_0 \;\leq\; R_N^E(\mu)^2 \;\leq\; C_1 \;<\; \infty \;,$$

which in turn imply

$$\rho([E_0, E_1]) \;\approx\; \lim_{L\to\infty} \int_{E_0}^{E_1} dE \int_{\mu_0}^{\mu_1} d\mu \; \frac{1}{(R_{L,N}^E(\theta_N^E(\mu)))^2} \;,$$

where the notation $f \approx g$ means that there are positive constants c_0, c_1 such that $c_0 f \leq g \leq c_1 f$, and the limit may not exist and is either the superior or inferior limit depending on whether one deals with the upper or lower bound. By hypothesis (i), one can make a change of variables in the μ-integral:

$$\rho([E_0, E_1]) \;\approx\; \lim_{L\to\infty} \int_{E_0}^{E_1} dE \int_{\theta(\mu_0)}^{\theta(\mu_1)} d\theta \; \left|\frac{d\mu}{d\theta}\right| \; \frac{1}{(R_{L,N}^E(\theta))^2} \;.$$

Now hypothesis (i) allows to bound the Jacobian from above and below, and then hypothesis (ii) allows to complete the proof using Proposition 5. □

This proposition cannot be used to rederive the classical spectral averaging over boundary conditions because the family $\alpha \mapsto H_\alpha$ is not differentiable at $\frac{\pi}{2}$. In our application, we consider Jacobi matrices of the form

$$H(\mu) \;=\; H(0) + \mu\, W \;, \qquad W \;=\; \sum_{n=1}^{N} w_n \, |n\rangle\langle n| \;,$$

where $H(0)$ is a given semi-infinite Jacobi matrix, $N < \infty$ and $w_1, \ldots, w_N \geq 0$ so that the potential W is positive. Averaging will be done over the parameter μ and the following results tell us under which conditions on the size of the interval $[\mu_0, \mu_1]$ the above proposition can be applied. One condition will be expressed in terms of the associated Birman-Schwinger operator (a self-adjoint $N \times N$ matrix) defined for any energy $E \in \mathbb{R}$ not in the spectrum of $H^N(0)$ by

$$K_E^N \;=\; W^{\frac{1}{2}} \, (E - H^N(0))^{-1} \, W^{\frac{1}{2}} \, ,$$

where $H^N(\mu)$ is the $N \times N$ matrix given by the upper left corner of $H(\mu)$. Let $E_1(\mu) < \ldots < E_N(\mu)$ denote the eigenvalues of $H^N(\mu)$ which are known to be all distinct.

Theorem 1 *Fix some $E \in \mathbb{R}$. Suppose that two consecutive values w_m, w_{m+1} are strictly positive and that one of the following conditions on the size of the interval $[\mu_0, \mu_1]$ hold:*
(a) *There are $\mu_0', \mu_1' \in (\mu_0, \mu_1)$ such that $E_n(\mu_0') = E_{n-1}(\mu_1') = E$ for some $n = 2, \ldots, N$.*
(b) *The potential W is strictly positive, E is not in the spectrum of $H^N(0)$ and there exist two non-vanishing eigenvalues $\lambda_0(E)$ and $\lambda_1(E)$ of K_E^N such that*

$$\mu_0 \;<\; \frac{1}{\lambda_0(E)} \;<\; \frac{1}{\lambda_1(E)} \;<\; \mu_1 \, . \tag{11}$$

Then the averaged spectral measure ρ defined as in (9) is equivalent to the Lebesgue measure in an open interval around E.

Proof. The monotonicity condition (i) of Proposition 8 holds due to Proposition 6 (ii) applied to all sites, in particular the two sites $m, m+1$ give the strict monotonicity because the wave function ϕ^E cannot vanish at two consecutive sites. Now each of the two hypothesis (a) and (b) imply the condition Proposition 8(ii). In the case (a) this follows immediately from the Sturm oscillation theorem (*e.g.* Section 3.2 of [9]), which states that the Prüfer phase $\theta_N^E(\mu)$ has to vary by more than π in $[\mu_0, \mu_1]$, namely $\theta_N^E(\mu_1) - \theta_N^E(\mu_0) > \pi$. In order to use (b), let us first rewrite the eigenvalue equation $H^N(\mu)\phi = E\phi$ as $(E - H^N(0))\phi = \mu W \phi$. As W and therefore $W^{\frac{1}{2}}$ are invertible, it hence follows that E is an eigenvalue of $H^N(\mu)$ if and only if $\frac{1}{\mu}$ is an eigenvalue of K_E^N with eigenstate $W^{\frac{1}{2}}\phi$. This will be used in order to analyze the eigenvalues of $H^N(\mu)$. If $I = [\mu_0, \mu_1]$ satisfies (11), then the interval I contains $\mu_0' < \mu_1'$ such that $E_n(\mu_0') = E$ and $E_{n-1}(\mu_1') = E$ for some n in $2, \ldots N$ and one can apply the condition (a). By continuity, all the above holds for an open interval containing E, so that Proposition 8 can be applied in order to conclude the proof. □

In order to show that condition (b) invoking the Birman-Schwinger operator can be more practical, let us treat an

Example: Let $N = 2$, $w_1, w_2 > 0$ and $H^2(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For $E \neq \pm 1$, one then has $K_E^2 = \frac{1}{E^2-1} \begin{pmatrix} Ew_1 & \sqrt{w_1 w_2} \\ \sqrt{w_1 w_2} & Ew_2 \end{pmatrix}$. The eigenvalues are

$$\lambda_\pm(E) \;=\; \frac{1}{4(E^2-1)} \left[\, E(w_1 + w_2) \pm \sqrt{E^2(w_1^2 + w_2^2) + 4w_1 w_2} \, \right] \, .$$

The condition (11) can be written out explicitly, the order of $\lambda_+(E)$ and $\lambda_-(E)$ therein depending on whether $|E| > 1$ or $|E| < 1$.

Remark The condition that two adjacent values w_m, w_{m+1} are strictly positive is needed in order to assure the monotonicity condition (i) in Proposition 8. This can be relaxed by asking, for example, that $\phi_m^E \neq 0$ for the smallest m such that $w_m > 0$. Note that ϕ_m^E then is independent of W.

Remark The situation where W has only one non-vanishing entry, say $w_N = 1$, can be dealt with in a manner similar to the average over the boundary condition, that is, one needs to average over $(\mu_0, \mu_1) = \mathbb{R}$ and then $\rho = \int_{\mathbb{R}} d\mu\, \rho_\mu$ dominates the Lebesgue measure on all $\mathbb{R}$. In order to show this, let us decompose the Prüfer radius as in the proof of Proposition 8 in $R_L^E = R_{L,N}^E(\theta_N^E) R_N^E$. Now $\cot(\theta_N^E) = E - \mu - (t_N^E)^2 \tan(\theta_{N-1}^E)$ and θ_{N-1}^E is independent of μ. Therefore, as μ varies in all $\mathbb{R}$, θ_N^E varies over all $[0, \pi)$ for all θ_{N-1}^E. Moreover, diagonalizing $R_\eta^* |\mathcal{T}^E(L, N)|^2 R_\eta = \mathrm{diag}(\kappa, 1/\kappa)$ with the adequate rotation R_η then shows

$$\int_{\mathbb{R}} d\mu\, \frac{1}{(R_{L,N}^E(\theta_N^E(\mu)))^2} \;=\; \int_{\mathbb{R}} d\mu\, \frac{1}{\kappa \cos^2(\theta_N^E + \eta) + \frac{1}{\kappa} \sin^2(\theta_N^E + \eta)}$$

The integral on the r.h.s. can be bounded below by a constant uniformly in θ_{N-1}^E because $\frac{d\mu}{d\theta} \geq 1$. This implies the claim as in the proof of Proposition 8.

4 Several-parameter spectral averaging

There are models for which several parameters are available for spectral averaging, but not any of them is sufficient by itself in order to lead to an averaged spectral measure which is equivalent to the Lebesgue measure. Then an averaging over several parameters may nevertheless allow to prove such a statement. In this section, we consider the concrete example of a semi-infinite Jacobi matrix of the form

$$H_{\lambda,N,v} \;=\; \Delta_N \,+\, \lambda \sum_{n=1}^{N} v_n\, |n\rangle\langle n| \,+\, J_N \;, \tag{12}$$

where Δ_N is the discrete Laplacian up to site N (namely, $t_n = 1$ and $v_n = 0$ for $n = 1, \ldots, N$, and vanishing coefficients afterwards), J_N is an arbitrary Jacobi matrix in the limit point case with $t_n = v_n = 0$ for $n = 1, \ldots, N$, $\lambda \geq 0$ is a coupling constant, and the entries of $v = (v_1, \ldots, v_N)$ are independent real random variables, each distributed according to the Lebesgue measure on $[-\frac{1}{2}, \frac{1}{2}]$. We also write $\mathbf{P}(dv)$ for this product measure on the unit cube $I_N = [-\frac{1}{2}, \frac{1}{2}]^{\times N}$. By our methods and a little more notational effort (which we choose to avoid), the model (12) could be generalized in order to allow for an arbitrary periodic background instead of Δ_N and arbitrary local perturbations on each periodicity interval. We use Dirichlet boundary conditions $\alpha = 0$ and suppress the argument α in all formulas below. The spectral measure of $H_{\lambda,N,v}$ w.r.t. $|1\rangle$ will be denoted by $\rho_{\lambda,N,v}$.

Theorem 2 *Let $0 < \lambda < 4$ and I be an open interval such that its closure is contained in $(-2+\frac{\lambda}{2}, 2-\frac{\lambda}{2})$. Then there exists an $N = N(\lambda)$ such that the averaged spectral measure*

$$\rho_{\lambda,N} \;=\; \int_{I_N} \mathbf{P}(dv)\; \rho_{\lambda,N,v}\;,$$

is equivalent to the Lebesgue measure in I.

This result is very similar to Wegner's upper and lower bound on the density of states for multi-dimensional discrete random Schrödinger operators [16, 11], however, our proof uses a different change of variables and is restricted to the one-dimensional situation due to the use of Prüfer variables. Our initial intent was to improve on Wegner's lower bound in the one-dimensional situation in the weak coupling limit. Indeed, the lower bound on the density of states as obtained in [16, 11] as well as ours goes to zero as λ goes to zero, which is absurd for energies in the spectrum (the integrated density of states varies only of order λ for all energies in the spectrum [9]). The problem behind this short-coming is that not sufficiently many random potential values are used for averaging (because the errors cannot be controled). As we will argue heuristically below, the N needed in Theorem 2 should actually be of order λ^{-2} (which is the localization length).

The Prüfer phases and radii at disorder configuration v and coupling constant $\lambda > 0$ are denoted by $\theta^E_{\lambda,n}(v)$ and $R^E_{\lambda,n}(v)$. The proof of Theorem 2 will use modified Prüfer variables (even though not in an optimized way as explained below). Let us recall their definition, *e.g.* from [12, 9]. For $E \in (-2,2)$ and $k = \arccos(E/2) \in (0, \frac{\pi}{2})$, one sets

$$M^E \;=\; \frac{1}{\sqrt{\sin(k)}} \begin{pmatrix} \sin(k) & 0 \\ -\cos(k) & 1 \end{pmatrix}\;.$$

Furthermore denote $e_\theta = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$. Now one defines a smooth function $m^E : \mathbb{R} \to \mathbb{R}$ with $m^E(\theta+\pi) = m^E(\theta) + \pi$ and $0 < C_1 \leq (m^E)' \leq C_2 < \infty$, by

$$r(\theta) e_{m^E(\theta)} = M^E e_\theta, \qquad r(\theta) > 0\;, \qquad m^E(0) \in [-\pi, \pi)\;.$$

Then the E-modified Prüfer variables $(\hat{R}^E_{\lambda,n}(v), \hat{\theta}^E_{\lambda,n}(v)) \in \mathbb{R}_+ \times \mathbb{R}$ for initial condition $\hat{\theta}^E_0 = m^E(\theta^E_0)$ are given by

$$\hat{\theta}^E_{\lambda,n}(v) \;=\; m^E(\theta^E_{\lambda,n}(v))\;, \tag{13}$$

and

$$\begin{pmatrix} \hat{R}^E_{\lambda,n}(v)\cos(\hat{\theta}^E_{\lambda,n}(v)) \\ \hat{R}^E_{\lambda,n}(v)\sin(\hat{\theta}^E_{\lambda,n}(v)) \end{pmatrix} \;=\; M^E \begin{pmatrix} t_n\,\phi^E_n \\ \phi^E_{n-1} \end{pmatrix}\;. \tag{14}$$

Important for our purposes are two facts (*e.g.* [9]). First of all, $|\hat{\theta}^E_{\lambda,n}(v) - \theta^E_{\lambda,n}(v)| \leq 2\pi$ for all $n \geq 0$. Second of all, the behavior of the E-modified Prüfer variables is very simple at $\lambda = 0$, namely

$$\hat{R}^E_{0,n}(v) = \hat{R}^E_{0,0}(v)\;, \qquad \hat{\theta}^E_{0,n}(v) = \hat{\theta}^E_0 + nk\;. \tag{15}$$

Proof of Theorem 2. The upper bound follows immediately as in Proposition 8, so we will only focus on the lower bound here. Let $E_0, E_1 \in I$. Proceeding exactly as in the proof of Proposition 8, one shows

$$\rho_{\lambda,N}([E_0,E_1]) \;=\; \lim_{L\to\infty} \int_{E_0}^{E_1} dE \int_{I_N} \mathbf{P}(dv)\, \frac{1}{R^E_{\lambda,L}(v)^2}\;.$$

The main advantage of this formula is that one can pass to E-modified Prüfer variables at every energy $E \in [E_0, E_1]$. Indeed, the Prüfer radii are changed at most by a factor which can be uniformly bounded in energy:

$$\rho_{\lambda,N}([E_0,E_1]) \;\geq\; C_0 \lim_{L\to\infty} \int_{E_0}^{E_1} dE \int_{I_N} \mathbf{P}(dv)\, \frac{1}{\hat{R}^E_{\lambda,L}(v)^2}\;.$$

Now we split into two contributions as in (10):

$$\rho_{\lambda,N}([E_0,E_1]) \;\geq\; C_0 \liminf_{L\to\infty} \int_{E_0}^{E_1} dE \int_{I_N} \mathbf{P}(dv)\, \frac{1}{\hat{R}^E_{\lambda,L,N}(\hat{\theta}^E_{\lambda,N}(v))^2}\, \frac{1}{\hat{R}^E_{\lambda,N}(v)^2}\;. \tag{16}$$

One may now use the positive constant (depending on N and λ, just as all the other constants below as well)

$$C_1 \;=\; \max_{E_0\leq E\leq E_1,\, v\in I_N} \hat{R}^E_{\lambda,N}(v)^2\;,$$

in order to bound the second factor of the integrand in (16):

$$\rho_{\lambda,N}([E_0,E_1]) \;\geq\; \frac{C_0}{C_1} \liminf_{L\to\infty} \int_{E_0}^{E_1} dE \int_{I_N} \mathbf{P}(dv)\, \frac{1}{\hat{R}^E_{\lambda,L,N}(\hat{\theta}^E_{\lambda,N}(v))^2}\;. \tag{17}$$

The strategy is to exhibit an adequate transformation of variables in I_N in order to be able to apply once again Proposition 5.

First let us analyze which values the modified Prüfer phase $\hat{\theta}^E_{\lambda,N}(v)$ in (17) may take. As it is monotonous in each v_n by Proposition 6(ii) and because m^E is a diffeomorphism, it follows that the smallest and largest values are $\hat{\theta}_0 = \hat{\theta}^E_{\lambda,N}(-\frac{1}{2},\ldots,-\frac{1}{2})$ and $\hat{\theta}_1 = \hat{\theta}^E_{\lambda,N}(\frac{1}{2},\ldots,\frac{1}{2})$. As the initial condition for the modified Prüfer variables is shifted by a term $\mathcal{O}(1)$ w.r.t. N, one now has $\hat{\theta}_0 = \hat{\theta}^{E+\frac{\lambda}{2}}_{0,N}(v) + \mathcal{O}(1)$ and $\hat{\theta}_1 = \hat{\theta}^{E-\frac{\lambda}{2}}_{0,N}(v) + \mathcal{O}(1)$, for arbitrary v. Using (15), it therefore follows that

$$\hat{\theta}_1 - \hat{\theta}_0 \;=\; N\left[\arccos((E-\frac{\lambda}{2})/2) - \arccos((E+\frac{\lambda}{2})/2)\right] \;+\; \mathcal{O}(1) \;\geq\; C_2 N\lambda \;+\; \mathcal{O}(1)\;,$$

so that one can choose N of order $1/\lambda$ such that $\hat{\theta}_1 - \hat{\theta}_0 > \pi$.

For every $N \geq 2$, $\|\nabla_v \hat{\theta}^E_{\lambda,N}(v)\| > 0$ by Proposition 6(ii) because the Prüfer radius is bounded and no eigenfunction can vanish at two consecutive sites. Therefore the map $v \in I_N \mapsto \hat{\theta}^E_{\lambda,N}(v)$ has no critical point and the sets

$$P^E_{\lambda,N}(\hat{\theta}) \;=\; \left\{\; v\in I_N \;\middle|\; \hat{\theta}^E_{\lambda,N}(v) = \hat{\theta}\;\right\}\;, \qquad \hat{\theta}_0 \leq \hat{\theta} \leq \hat{\theta}_1\;. \tag{18}$$

are (real analytic) subvarieties of I_N of co-dimension 1, with boundaries of co-dimension 2. Then $[\hat{\theta}_0, \hat{\theta}_1]$ is precisely the interval of $\hat{\theta}$'s for which $P^E_{\lambda,N}(\hat{\theta})$ is not empty. Because the gradient (w.r.t. v) of $\hat{\theta}^E_{\lambda,N}(v)$ does not vanish, $P^E_{\lambda,N}(\hat{\theta}_0)$ and $P^E_{\lambda,N}(\hat{\theta}_1)$ consist of only one point each and Morse theory implies that all other manifolds $P^E_{\lambda,N}(\hat{\theta})$, $\hat{\theta} \in (\hat{\theta}_0, \hat{\theta}_1)$, are diffeomorphic [7, Theorem 6.2.2]. This implies also that the $N-1$-dimensional volume measured with the $N-1$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$ of the hyper-surfaces $P^E_{\lambda,N}(\hat{\theta})$, $\hat{\theta} \in (\hat{\theta}_0, \hat{\theta}_1)$, is positive. Therefore there exists a constant $C_3 > 0$ such that $\mathcal{H}^{N-1}(P^E_{\lambda,N}(\hat{\theta})) \geq C_3$ for all $\hat{\theta}$ in the smaller interval $[\hat{\theta}'_0, \hat{\theta}'_1] \subset (\hat{\theta}_0, \hat{\theta}_1)$. This can be done such that $\hat{\theta}'_1 - \hat{\theta}'_0 > \pi$.

The change of variable will now be based on Federer's coarea formula [6]. In the situation relevant for our purposes, it states that for all Lipshitz continuous functions g

$$\int_{I_N} \mathbf{P}(dv)\; J_1(\hat{\theta}^E_{\lambda,N}(v))\; g(\hat{\theta}^E_{\lambda,N}(v)) \;=\; \int_{\hat{\theta}_0}^{\hat{\theta}_1} d\hat{\theta}\; \mathcal{H}^{N-1}(P^E_{\lambda,N}(\hat{\theta}))\; g(\hat{\theta})\;,$$

where the 1-Jacobian is given by

$$J_1(\hat{\theta}^E_{\lambda,N}(v)) \;=\; \left\|\nabla_v \hat{\theta}^E_{\lambda,N}(v)\right\| \;=\; \left(\sum_{n=1}^{N} |\partial_{v_n} \hat{\theta}^E_{\lambda,N}(v)|^2\right)^{\frac{1}{2}}\;.$$

By compactness, one has $J_1(\hat{\theta}^E_{\lambda,N}(v)) \leq C_4$ for all $v \in I_N$ and $E \in [E_0, E_1]$. Now using the coarea formula for $g(\hat{\theta}) = \hat{R}^E_{\lambda,L,N}(\hat{\theta})^{-2}$ gives

$$\begin{aligned}
\int_{I_N} \mathbf{P}(dv) \frac{1}{\hat{R}^E_{\lambda,L,N}(\hat{\theta}^E_{\lambda,N}(v))^2} \;&\geq\; \frac{1}{C_4} \int_{I_N} \mathbf{P}(dv)\; J_1(\hat{\theta}^E_{\lambda,N}(v))\; \frac{1}{\hat{R}^E_{\lambda,L,N}(\hat{\theta}^E_{\lambda,N}(v))^2} \\
&=\; \frac{1}{C_4} \int_{\hat{\theta}_0}^{\hat{\theta}_1} d\hat{\theta}\; \mathcal{H}^{N-1}(P^E_{\lambda,N}(\hat{\theta}))\; \frac{1}{\hat{R}^E_{\lambda,L,N}(\hat{\theta})^2} \\
&\geq\; \frac{C_3}{C_4} \int_{\hat{\theta}'_0}^{\hat{\theta}'_1} d\hat{\theta}\; \frac{1}{\hat{R}^E_{\lambda,L,N}(\hat{\theta})^2}\;.
\end{aligned}$$

As $\hat{\theta}'_1 - \hat{\theta}'_0 > \pi$, the full projective space is covered. The Möbius transformation with M^E does not change this property. Thus one may pass back to non-modified Prüfer variables at the cost of another constant. Therefore the r.h.s. can then be bounded below by a positive constant independent of L due to Proposition 5. Replacing this bound in (17) completes the proof. □

We conclude this paragraph with some heuristics as to how the above proof can be modified in order to yield an improvement of Wegner's lower bound [16, 11]. More precisely, we shall argue that one should be able to choose $N = N(\lambda) = C_5\lambda^{-2}$ in Theorem 2 for some adequate constant C_5 and that for this choice the lower bound on the averaged spectral measure remains positive in the limit $\lambda \to 0$. The choice $N = C_5\lambda^{-2}$ means that one is precisely at the scale of the localization length [12, 9]. Thus the norm of the transfer matrices still has not begun to grow exponentially, and therefore $\hat{R}^E_{\lambda,N}(v) = \mathcal{O}(1)$ with high

probability in v w.r.t. $\mathbf{P}$. If one supposes that this holds uniformly in v (which is wrong, of course, and would have to be replaced by a probabilistic argument), one hence has $C_1 = \mathcal{O}(1)$. As C_0 is independent of N and λ, this allows to start from (17) with constants of order of unity. Now due to the choice of $N(\lambda)$, one has $\hat{\theta}_1 - \hat{\theta}_0 = \mathcal{O}(\lambda^{-\frac{1}{2}})$ by the same calculation as above, which is much larger than the one turn needed in order to conclude the argument. However, the $N-1$-dimensional volume of $P^E_{\lambda,N}(\hat{\theta})$ is very small for most $\hat{\theta} \in [\hat{\theta}_0, \hat{\theta}_1]$ and one has to select those for which it is of order of unity. For this purpose, let us recall from [9] that the modified Prüfer phases can be expanded as follows

$$\hat{\theta}^E_{\lambda,N}(v) \;=\; \hat{\theta}^E_0 \;+\; N\,k \;+\; \lambda \sum_{n=1}^{N} v_n \left[1 + \cos(2\hat{\theta}^E_{\lambda,n-1}(v))\right] \;+\; \mathcal{O}(N\lambda^2)\,. \tag{19}$$

As the v_n are centered and $N = C_5\lambda^{-2}$, the central limit theorem implies that $\lambda \sum_{n=1}^{N} v_n$ converges in the limit $\lambda \to 0$ in distribution to a centered Gaussian. Hence this term is of order 1 with positive probability. On the other hand, the sum $\lambda \sum_{n=1}^{N} v_n \cos(2\hat{\theta}^E_{\lambda,n-1}(v))$ is expected to be of order λ for almost all v. Neglecting this term as well as the error term $\mathcal{O}(N\lambda^2)$ in (19) then leads to

$$P^E_{\lambda,N}(\hat{\theta}) \;\approx\; \left\{ \; v \in I_N \;\middle|\; \hat{\theta} = \hat{\theta}^E_0 \;+\; N\,k \;+\; \lambda \sum_{n=1}^{N} v_n \; \right\}\,. \tag{20}$$

Now we choose $\hat{\theta}'_0 = \hat{\theta}^E_0 + Nk - C_6$ and $\hat{\theta}'_1 = \hat{\theta}^E_0 + Nk + C_6$ for some $C_6 > 0$ sufficiently large. By (20) and the central limit theorem, one then has $\mathcal{H}^{N-1}(P^E_{\lambda,N}(\hat{\theta})) \geq C_7$ for all $\hat{\theta} \in [\hat{\theta}'_0, \hat{\theta}'_1]$. Using that also the 1-Jacobian is of order 1 (which can roughly be deduced from (19)), this should allow to conclude the proof just as above. The lower bound obtained is then independent of λ.

5 Applications to spectral analysis

The two applications of this section both concern the model (12), but the first one is based on an application also of Theorem 1. Because the appearing J_N is arbitrary, the operators $H_{\lambda,N,v}$ can have all spectral types. We state the following propositions for the singular part of the spectrum, but it holds for many other case (see the Remark below).

Proposition 9 *Let us fix $\hat{\lambda} \in (0,4)$ as well as a corresponding interval $\hat{I}$ and integer $\hat{N} = N(\hat{\lambda})$ as given in* Theorem 2. *Furthermore, fix some $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_{\hat{N}})$ with $\hat{v}_n \geq 0$ and two adjacent strictly positive entries. Then the following two statements are equivalent:*

(i) *$H_{\lambda,\hat{N},\hat{v}}$ has singular spectrum in $\hat{I}$ for all $\lambda \in B$ where $B \subset \mathbb{R}$ is of positive Lebesgue measure, namely $|B| > 0$.*

(ii) *$H_{\hat{\lambda},\hat{N},v}$ has singular spectrum in $\hat{I}$ for all $v \in D \subset I_{\hat{N}}$ where $\mathbf{P}(D) > 0$.*

Proof. (i) $\Rightarrow$ (ii): Let $S \subset \mathbb{R}$ be the set of energies for which subordinate solutions of $H_{\lambda,\hat{N},\hat{v}}$ exist. This set is a support of the singular part and is independent of $\lambda, \hat{N}, \hat{v}$. If

$H_{\lambda,\hat{N},\hat{v}}$ has singular spectrum in $\hat{I}$ for $\lambda \in B$ where $|B| > 0$, then one has $\rho_{\lambda,\hat{N},\hat{v}}(S \cap \hat{I}) > 0$ for all $\lambda \in B$ (see *e.g.* by Corollary 2.8 of [4]). This implies that $\int_{\mathbb{R}} d\lambda\, \rho_{\lambda,\hat{N},\hat{v}}(S \cap \hat{I}) > 0$. Therefore Theorem 1 implies that $|S \cap \hat{I}| > 0$. This in turn implies by Theorem 2 that $\int_{I_N} \mathbf{P}(dv)\, \rho_{\hat{\lambda},\hat{N},v}(S \cap \hat{I}) > 0$ so that $\rho_{\hat{\lambda},\hat{N},v}(S \cap \hat{I}) > 0$ for all $v \in D$ with $\mathbf{P}(D) > 0$. Hence again $H_{\hat{\lambda},\hat{N},v}$ has singular spectrum in $\hat{I}$ for $v \in D$ with $\mathbf{P}(D) > 0$.

(ii) $\Rightarrow$ (i): If $H_{\lambda,\hat{N},\hat{v}}$ has singular spectrum for a set of v's of positive measure, then $\int_{I_N} \mathbf{P}(dv)\, \rho_{\hat{\lambda},\hat{N},v}(S \cap \hat{I}) > 0$ so that $|S \cap \hat{I}| > 0$ by Theorem 2. For λ_0 and λ_1 adequately chosen, Theorem 1 then implies that $\int_{\lambda_0}^{\lambda_1} d\lambda\, \rho_{\lambda,\hat{N},\hat{v}}(S \cap \hat{I}) > 0$, which shows (i). □

By the same proof evoking either Theorem 1 or Theorem 2 twice (and nowhere the other) one proves the following results:

Proposition 10 *Let us fix an open interval $\hat{I}$ and integers $\hat{N}_0, \hat{N}_1$ as well as positive $\hat{v}_0 \in I_{\hat{N}_0}, \hat{v}_1 \in I_{\hat{N}_1}$. Then the following two statements are equivalent:*

(i) *$H_{\lambda,\hat{N}_0,\hat{v}_0}$ has singular spectrum in $\hat{I}$ for a set of λ's of positive Lebesgue measure.*

(ii) *$H_{\lambda,\hat{N}_1,\hat{v}_1}$ has singular spectrum in $\hat{I}$ for a set of λ's of positive Lebesgue measure.*

Proposition 11 *Let us fix $\hat{\lambda}_0, \hat{\lambda}_1 \in (0,4)$, and corresponding to both as in* Theorem 2 *an interval $\hat{I}$ and an integer $\hat{N} = N(\hat{\lambda})$. Then the following two statements are equivalent:*

(i) *$H_{\hat{\lambda}_0,\hat{N},v}$ has singular spectrum in $\hat{I}$ for all $v \in D_0 \subset I_{\hat{N}}$ where $\mathbf{P}(D_0) > 0$.*

(ii) *$H_{\hat{\lambda}_1,\hat{N},v}$ has singular spectrum in $\hat{I}$ for all $v \in D_1 \subset I_{\hat{N}}$ where $\mathbf{P}(D_1) > 0$.*

Remark: The same results hold for the pure-point part of the spectrum and its singular continuous part, as well as for α-continuity and α-singularity of the spectral measures (for the proof of the latter, combine the above arguments with those of [8, 10].). With some care, it is possible to further localize the set B in Proposition 9 in $(0, 2)$.

With a similar proof working with zero measure sets, one can also obtain results analogous to the above propositions. We have, for example, the following:

Proposition 12 *Under the same hypothesis as in* Proposition 9*, the following two statements are equivalent:*

(i) *$H_{\lambda,\hat{N},\hat{v}}$ has pure-point spectrum in $\hat{I}$ for Lebesgue almost all $\lambda \in \mathbb{R}$.*

(ii) *$H_{\hat{\lambda},\hat{N},v}$ has pure-point spectrum in $\hat{I}$ for $\mathbf{P}$-almost all $v \in I_{\hat{N}}$.*

It seems to be an open question whether this proposition is true when the word "pure" is omitted.

Acknowledgment: R. del R. thanks B. Simon for pertinent hints to the literature. Our work and respective visits were supported by PROYECTO PAPIIT IN-111906 UNAM and by the DFG. C. M. was funded by CONACYT.

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