W.O. Amrein, A.M. Hinz, D.B. Pearson
Sturm-Liouville Theory: Past and Present, 217–235

Boundary Conditions and Spectra of Sturm-Liouville Operators

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Abstract. This is a discussion of some aspects of the relation between boundary conditions and spectra of Sturm-Liouville operators. It is intended to review results which show how the spectrum behaves when the boundary condition changes. The absolutely continuous part will normally be stable and the more interesting problems concern the behavior of the singular part and coexistence of different spectral types.

1. Introduction

This paper is a survey of some aspects of the relation between boundary conditions and spectrum of Sturm-Liouville operators. The main problem is to understand how the spectrum behaves when the boundary condition varies. In particular it is of interest to study the behavior of the different parts of the spectrum (for example singular and absolutely continuous). These operators were introduced in [38, 39].

A key tool of some developments I intend to describe is the so-called Weyl m-function. This function is analytic in the upper half-plane and closely connected to the resolvent of the operator. Its study will allow us to clarify what happens with the spectrum when the boundary condition changes.

In 1910 H. Weyl proved that the essential spectrum, which in this case is just the set of accumulation points of the spectrum, is stable when the boundary condition is modified. What changes in fact are the isolated points. In 1957 several remarkable papers were published. M. Rosenblum [35] and T. Kato [23] proved stability of absolutely continuous spectra for self-adjoint operators under trace class perturbations and N. Aronszajn [1] showed that the absolutely continuous parts of spectral measures of Sturm-Liouville problems corresponding to different boundary conditions are equivalent[1], whereas their singular parts are mutually

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Partially supported by Project 37444E CONACyT.
[1] *i.e.*, they lead to the same null sets.

singular measures[2]. To the best of my knowledge this is the first study of singular continuous spectra of differential operators.

Since the absolutely continuous part is stable, it is of particular interest to understand the behavior of the singular part, especially that part which is embedded in the essential spectrum. As was already noticed by Aronszajn, this part is very unstable. In fact it is not possible for all boundary conditions to allow eigenvalues embedded in the essential spectrum. This kind of spectra will have to disappear when the boundary condition changes. We can nevertheless have singular spectrum which is embedded in the absolutely continuous spectrum, for all boundary conditions. In this review the above-mentioned result of Aronszajn will be described and several other theorems which clarify to some extent the behavior of the embedded singular part will be sketched. I will concentrate mainly on results that are more familiar to me.

After a brief description of the result on stability proven by Aronszajn, a theorem due to Hartman and Wintner on the behavior of isolated eigenvalues is stated. This result mainly says that isolated eigenvalues, that is eigenvalues that are in gaps of the essential spectrum, behave smoothly under the considered perturbations and that the complement of the essential spectrum is contained in the interior of the set of points $\lambda \in \mathbb{R}$ for which there are L^2 solutions of $-u'' + v(x)u = \lambda u$.

Following this, the problem of embedded singular spectrum is considered. The basic tools used are properties of the Weyl m-function and some results of [1]. It is shown that coexistence of singular and absolutely continuous spectrum is possible for large sets of boundary conditions. Here is explained in more detail a result that gives conditions on the length of an interval for the parameter of boundary conditions which imply that this interval contains a set of full measure where singular and absolutely continuous spectra coexist.

Thereafter, an example of a very explicit spectral function is given which generates a situation with mixed spectra for all boundary conditions with the exception of one. Using the inverse spectral theorem of Gelfand-Levitan it is known that Sturm-Liouville operators with this kind of spectral function exist, and therefore that mixed situations even for large sets of boundary conditions are possible. It remains to carry out an explicit construction of such operators.

Finally some results on inverse spectral theory of regular problems are considered. If the spectra are known for two boundary conditions, the Sturm-Liouville operator can be uniquely reconstructed [2] (see also the article by M. Malamud in this volume). It happens that if we know something about the potential then we need less information about the spectra. This kind of problem is very different from those considered above and illustrates the role of the relation between boundary conditions and spectra in other settings.

I hope this text will be useful to those wishing to understand the important relations between boundary conditions and spectra of Sturm-Liouville operators.

[2] *i.e.*, they are concentrated on mutually disjoint sets of Lebesgue measure zero.

2. Some classical results

We consider one-dimensional Schrödinger equations

$$\mathcal{L}y = -y''(x) + v(x)\,y(x) = \lambda y(x), \quad 0 \le x < \infty, \tag{1}$$

and the associated self-adjoint operators

$$H_\alpha = -\frac{d^2}{dx^2} + v(x) \quad \text{in } L^2(0,\infty)$$

generated by the boundary condition

$$y(0)\cos\alpha - y'(0)\sin\alpha = 0, \quad \alpha \in [0,\pi). \tag{2}$$

Here we assume that the real function v is locally of class L^1 on $[0,\infty)$ and that the limit point case holds at ∞.

Let $u_1(x,z)$ and $u_2(x,z)$ be solutions of

$$\mathcal{L}u = zu \tag{3}$$

which satisfy

$$\begin{aligned} u_1(0,z) &= \sin\alpha, & u_1'(0,z) &= \cos\alpha, \\ u_2(0,z) &= -\cos\alpha, & u_2'(0,z) &= \sin\alpha. \end{aligned}$$

For every non-real z there exists a function

$$\varphi_\alpha(x,z) = u_2(x,z) + m_\alpha(z)u_1(x,z)$$

which is a solution of (3) and belongs to $L^2(0,\infty)$. Note that u_1 satisfies the boundary condition (2). In the limit point case at ∞ (see [41]), for each z the complex number $m_\alpha(z)$ is defined uniquely. This is called the Weyl m-function for the boundary condition (2) given by α and has an integral representation of the form

$$m_\alpha(z) = c + \int_{\mathbb{R}} \left(\frac{1}{\mu - z} - \frac{\mu}{\mu^2+1}\right) d\rho_\alpha(\mu), \tag{4}$$

where ρ_α is a Lebesgue-Stieltjes measure uniquely determined by m_α. The measure $d\rho_\alpha$ is called the spectral measure, and ρ_α is called the spectral function, of the operator H_α. We shall denote by $\rho_\alpha(S)$ the spectral measure of a set S.

The spectral density $d\rho_\alpha/d\lambda$ is given almost everywhere by

$$\frac{d\rho_\alpha(\lambda)}{d\lambda} = \lim_{E\to 0+} \frac{1}{\pi}\mathrm{Im}\big(m_\alpha(\lambda + iE)\big) =: \frac{1}{\pi}\mathrm{Im}\big(m_\alpha(\lambda + i0)\big),$$

and may be thought of as a local probability density for the energy of the system.

Once we have a family of operators H_α depending on a parameter $\alpha \in \mathbb{R}$, it is quite natural to ask what happens when α varies. Are all H_α the same at least in some sense? How do properties or objects associated with H_α behave as we change α? H. Weyl in [41, 42] (*cf.* also [3]) proved that the essential spectrum of H_α, denoted by σ_{ess}, that is the set of points of accumulation of the spectrum, is independent of α and that point spectra corresponding to two different boundary conditions do not intersect. In the same paper Weyl states that he could not be sure that a similar stability result would hold for the continuous spectra.

It was N. Aronszajn who solved this problem in [1]. Since the methods used in his paper were the key to later developments, I shall try to explain their main aspects.

First note that the following relation holds for m:

$$m_\alpha(z) = \frac{m_\beta(z)\cos(\alpha-\beta) - \sin(\alpha-\beta)}{m_\beta(z)\sin(\alpha-\beta) + \cos(\alpha-\beta)}\,. \tag{5}$$

Once we have (5), we would like to know how the various spectral measures ρ_α are related, since spectral information about H_α is contained in the corresponding ρ_α.

In [1] minimal supports M_{ac}, M_s, M_{sc} and M_p of the absolutely continuous, singular, singular continuous and point parts of ρ_α, denoted respectively by $\rho_\alpha^{ac}, \rho_\alpha^s, \rho_\alpha^{sc}$ and ρ_α^p, are given as follows

$$\begin{aligned} M_{ac}^\alpha &= \{x \in E \mid 0 < \operatorname{Im}\big(m_\alpha(x+i0)\big) < \infty\}, \\ M_s^\alpha &= \{x \in E \mid \operatorname{Im}\big(m_\alpha(x+i0)\big) = \infty\}, \\ M_{sc}^\alpha &= \{x \in E \mid \operatorname{Im}\big(m_\alpha(x+i0)\big) = \infty,\ \rho_\alpha\{x\} = 0\}, \\ M_p^\alpha &= \{x \in E \mid \operatorname{Im}\big(m_\alpha(x+i0)\big) = \infty,\ \rho_\alpha\{x\} > 0\}, \end{aligned}$$

where

$$\operatorname{Im}\big(m_\alpha(x+i0)\big) := \lim_{y\downarrow 0} \operatorname{Im}\big(m_\alpha(x+iy)\big)$$

and $E = \{x \mid \operatorname{Im}\big(m_\alpha(x+i0)\big)$ exists$\}$. Using (5) one observes that

$$M_{ac}^\alpha = M_{ac}^\beta, \text{ and } M_s^\alpha \cap M_s^\beta = \varnothing \text{ for } \alpha \neq \beta.$$

This implies that the absolutely continuous parts of ρ_α are equivalent for all α, whereas their singular parts are mutually singular.

I should mention a remarkable paper due to D. Gilbert and D. Pearson [17] where the notion of subordinate solution is introduced and the above-mentioned supports are related to the behavior of solutions of the Schrödinger equation near the end points of the interval. See the contribution of D. Gilbert in this volume [16].

Since we have stability for the absolutely continuous part, the natural questions which arise concern the behavior of the singular part. For the singular part in the complement of the essential spectrum the following result holds (see [19], [20], [3] or [14, Theorem 2.5.3]).

Theorem 1. *Let the open interval I be a gap in σ_{ess} and let λ be a point in I. Then there is a unique α such that $\lambda \in \sigma_{\alpha\mathrm{d}}$. Writing α as $\alpha(\lambda)$, α can be taken to be a continuous increasing function of λ in I.*

Here $\sigma_{\alpha\mathrm{d}}$ denotes the set of isolated eigenvalues of H_α. In particular, from this result it follows that $\complement\sigma_{ess} \subset S_0$ where S_0 is the interior of the set

$$S = \{\lambda \in \mathbb{R} \mid \exists u \text{ a solution of } \mathcal{L}u = \lambda u \text{ such that } \int_0^\infty |u(t)|^2 dt < \infty\}.$$

The problem whether the sets $\complement\sigma_{ess}$ and S_0 are always equal was first studied in [20] and it was shown in [6], [32] that equality does not always hold. However, if the spectrum is a perfect set, that is, equal to the set of its limit points, then $S_0 = \complement\sigma_{ess}$. See [6].

3. Embedded singular spectrum

Now we turn to the study of the singular part which is embedded in the essential spectrum. Since the supports of the singular parts corresponding to different boundary conditions are mutually disjoint, we cannot expect much stability of this part. We already saw that the way isolated eigenvalues move when the boundary condition varies is very smooth. No such smoothness occurs for the embedded singular part.

Eigenvalues embedded in the essential spectrum may "live" only in a set of first category in the sense of Baire. In fact this is a general statement when we talk of supports of the various spectral measures ρ_α. Let I be an interval such that the spectrum is essentially dense in I, meaning that $\rho_0(J) > 0$ for every subinterval J of I. Then there exists a set F of first category which supports each of the measures ρ_α (not just the point part), i.e., such that $\rho_\alpha(I \backslash F) = 0$ for all α. See [11].

A basic tool for the understanding of the behavior of embedded eigenvalues is the following theorem of Aronszajn [1].

Theorem 2. *Consider the Sturm-Liouville equation* (1) *and two different boundary conditions corresponding to* $\alpha \neq \beta \bmod \pi$. *In order that* ξ *be in the point spectrum relative to the boundary condition* β, *it is necessary and sufficient that* $\int_{\mathbb{R}} (\lambda - \xi)^{-2} d\rho_\alpha(\lambda) < \infty$ *and that* $m_\alpha(\xi) + \cot(\beta - \alpha) = 0$.

Let us define $G(\xi) = \int_{\mathbb{R}}(\lambda-\xi)^{-2} d\rho_0(\lambda)$. In [10] it was proven that $\{y \mid G(y) = \infty\}$ is a dense G_δ set in supp $(d\rho_0)$, the support of $d\rho_0$. (Remember that a G_δ is a set which is a countable intersection of open sets.)

If we assume that an interval is contained in the spectrum, then the complement of $\{y \mid G(y) = \infty\}$ cannot contain this interval; moreover the support of the point part embedded in the spectrum has to be small in Baire sense, that is of first category.

Now we can use properties of m_0 to map this set of first category in the spectrum to a set of first category in the boundary conditions to obtain the following theorem [10].

Theorem 3. *The set* $\{\alpha \mid H_\alpha$ *has no eigenvalues in the spectrum of* $H_0\}$ *is a dense* G_δ *in* $[0, \pi]$.

Therefore the set of α for which H_α may have embedded eigenvalues is of first category in $[0, \pi]$. This theorem tells us that dense point spectra are very unstable and that even a very small perturbation of the boundary condition will make the whole point part disappear.

In some sense the essential spectrum prevents the existence of point spectra for many boundary conditions. An example where the above applies is given by the operator $H = d^2/dx^2 + \cos(\sqrt{x})$ on $L^2(0,\infty)$.

It was shown that for any boundary condition α the spectrum of the operator is absolutely continuous on $(1,\infty)$ [37] and that H_α has pure point spectrum in [-1,1] for a.e. α [24]. From what was mentioned above, it follows that for a dense G_δ of α, H_α has only singular continuous spectrum in $[-1,1]$. An open problem is to exhibit a α where this happens.

In 1993, N. Makarov [27] made the conjecture that

$$|\{\alpha \mid H_\alpha \text{ has only p.p. spectrum}\}| \cdot |\{\alpha \mid H_\alpha \text{ has only s.c. spectrum}\}| = 0\ ;$$

here and in the sequel, $|\cdot|$ denotes Lebesgue measure. As far as I know this remains an open question. In this context, natural questions arise about the possibility of coexistence of different types of spectra for large sets of boundary conditions α, in particular the coexistence of absolutely continuous and singular spectra. It is known that if the potential v is in L^2, then there can be singular spectrum at positive energies only for a set of boundary conditions of measure zero. This follows from a result of [4] which states that the support of the singular part in this case has Lebesgue measure zero and formula (7) below.

As mentioned above, the singular spectrum may be very unstable. Nevertheless, the property of having singular spectra for a set of boundary conditions of positive measure is preserved under L^1 perturbations to the potential; see [25], [12]. It is also worth mentioning that the exact Hausdorff dimension of the spectral measures may be the same for all boundary conditions, at least in the discrete case. See Theorem 4.3 in the contribution of Y. Last to this volume [26].

4. Sketch of a result on coexistence

It is possible to have absolutely continuous spectrum for all boundary conditions and singular spectrum for some boundary conditions. In [33] an example was constructed where for a set of boundary conditions of positive measure there is singular spectrum supported on a Cantor type set. This construction can be modified to have singular spectrum supported on a dense set (see [40]). In this example there is not much information about the set of boundary conditions with mixed spectra other than that this set is of positive measure. In what follows I shall sketch a result (see [13]) which gives a clearer idea of this set.

The following equality holds

$$\int_\alpha^\beta \rho_\theta(A)d\theta = \frac{1}{\pi}\int_A \arg\left[\frac{\cos\beta + \sin\beta\, m_0(\lambda + i0)}{\cos\alpha + \sin\alpha\, m_0(\lambda + i0)}\right] d\lambda, \tag{6}$$

which is a generalization of the well-known result, (see [36]):

$$\int_0^\pi \rho_\theta(A)d\theta = |A|. \tag{7}$$

Let us define $\Lambda_M := \{\lambda \mid \operatorname{Im}\big(m_0(\lambda+i0)\big) > M\}$. Then we have the following bound:

Lemma 1.

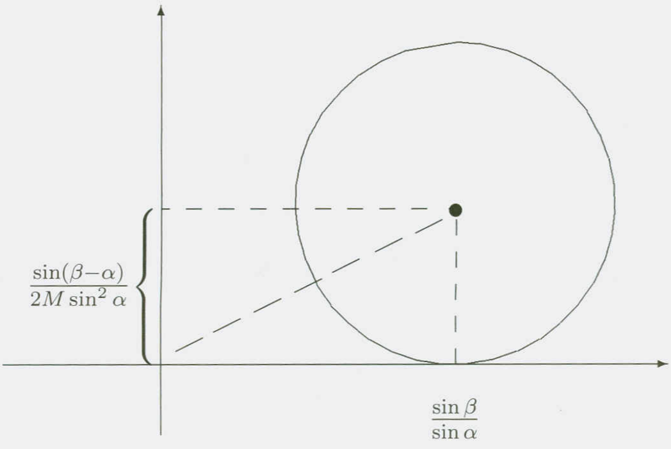
$$\int_\alpha^\beta \rho_\theta\Big(I\cap\Lambda_M\Big)d\theta \le \frac{2}{\pi}\arctan\Big(\frac{1}{2M}(\cot\alpha-\cot\beta)\Big)\Big|I\cap\Lambda_M\Big|.$$

Proof. The statement of the lemma follows if we put together equality (6) and the definition of Λ_M. Observe that the transformation

$$w = Tz = \frac{z\sin\beta+\cos\beta}{z\sin\alpha+\cos\alpha}$$

maps the half-plane $\operatorname{Im} z > M$ onto the disk (see the figure)

$$\left(x-\frac{\sin\beta}{\sin\alpha}\right)^2+\left(y-\frac{\sin(\beta-\alpha)}{2M\sin^2\alpha}\right)^2<\left(\frac{\sin(\beta-\alpha)}{2M\sin^2\alpha}\right)^2;$$



therefore $\operatorname{Im}\big(m_0(\lambda+i0)\big) > M$ implies that

$$\arg T(m_0(\lambda+i0)) \le 2\arctan\Big(\frac{1}{2M}(\cot\alpha-\cot\beta)\Big).$$

Using this and (6) we obtain

$$\begin{aligned}\int_\alpha^\beta \rho_\theta\Big(I\cap\Lambda_M\Big) &= \frac{1}{\pi}\int\limits_{I\cap\Lambda_M} \arg\Big(T(m_0(\lambda+i0)\Big)d\lambda\\ &\le \frac{2}{\pi}\int\limits_{I\cap\Lambda_M} \arctan\Big(\frac{1}{2M}(\cot\alpha-\cot\beta)\Big)d\lambda\\ &= \frac{2}{\pi}\arctan\Big(\frac{1}{2M}(\cot\alpha-\cot\beta)\Big)\Big|\Lambda_M\cap I\Big|.\end{aligned}$$

□

We also need a bound from below, for which the following theorem [34] will be useful.

Theorem 4. *Let λ_i, λ_j be eigenvalues of*

$$-y''(x) + v(x)y(x) = \lambda y(x) \qquad x \in [0, N],$$
$$y(0)\cos\alpha - y'(0)\sin\alpha = 0,$$
$$y(N)\cos\beta - y'(N)\sin\beta = 0.$$

Let $d\tilde{\rho}$ be the spectral measure of this problem. Then

$$\tilde{\rho}\left((\lambda_i, \lambda_j)\right) = \min_{\rho \in M_N} \rho\left((\lambda_i, \lambda_j)\right).$$

Here M_N is a family of measures which contains the spectral measure of the half-line problem. The bound from below that we need is given by the following lemma.

Lemma 2. *Let N be a positive real number and assume that $v(x) = 0$ for all $x \in [0, N]$. Set $I = \left(\left(\frac{\pi k}{N}\right)^2, \left(\frac{\pi(k+2)}{N}\right)^2\right)$ where $k \in \mathbb{N}$; then*

$$\int_\alpha^\beta \rho_\theta(I)d\theta \geq \frac{2\pi(k+1)}{N^2} \int_{\frac{N}{\pi(k+1)}\cot\beta}^{\frac{N}{\pi(k+1)}\cot\alpha} \frac{dx}{1+x^2}.$$

Proof. The function

$$\psi_\alpha\left(x, \frac{\pi k}{N}\right) = \frac{1}{2}\left[\sin\alpha + \frac{N}{i\pi k}\cos\alpha\right] \mathrm{e}^{\frac{i\pi k}{N}x} + \frac{1}{2}\left[\sin\alpha - \frac{N}{i\pi k}\cos\alpha\right] \mathrm{e}^{\frac{-i\pi k}{N}x}$$

is a solution of

$$-\psi''(x) = \left(\frac{\pi k}{N}\right)^2 \psi(x),$$
$$\psi(0)\cos\alpha - \psi'(0)\sin\,\alpha = 0,$$
$$\psi(N)\cos\alpha - \psi'(N)\sin\,\alpha = 0.$$

Observe that we have the same eigenvalues $\left(\frac{\pi k}{N}\right)^2$ for all $\alpha \in [0, \pi)$. Since

$$\|\psi_\alpha\|^2 = \int_0^N |\psi_\alpha(x)|^2 dx = \frac{(\sin\alpha)^2}{2}N + \frac{1}{2}N\left(\frac{N\cos\alpha}{\pi k}\right)^2$$

we get from Theorem 4:

$$\rho_\theta(I) \geq \tilde{\rho}(I) = \left\|\psi_\theta\left(x, \frac{\pi(k+1)}{N}\right)\right\|^{-2},$$

$$\int_\alpha^\beta \rho_\theta(I)d\theta \geq \int_\alpha^\beta \|\psi_\theta\|^{-2}d\theta = \frac{2\pi(k+1)}{N^2}\int_{\frac{N}{\pi(k+1)}\cot\beta}^{\frac{N}{\pi(k+1)}\cot\alpha} (1+x^2)^{-1}dx. \qquad \square$$

If we put together the bounds from above and below we obtain the following result:

Theorem 5. *Let $N \in \mathbb{R}^+, k \in \mathbb{N}$, and assume that $v(x) = 0$ for all $x \in [0, N]$. Set $I = \left(\left(\frac{\pi k}{N}\right)^2, \left(\frac{\pi(k+2)}{N}\right)^2 \right)$ and let $0 < \alpha < \beta < \pi$. If the following inequality is satisfied:*

$$\frac{2\pi(k+1)}{N^2} \int_{\frac{N}{\pi(k+1)}\cot\beta}^{\frac{N}{\pi(k+1)}\cot\alpha} \frac{dx}{1+x^2} > \frac{2|\Lambda_M \cap I|}{\pi} \int_0^{\frac{1}{2M}(\cot\alpha - \cot\beta)} \frac{dx}{1+x^2},$$

then

$$\int_\alpha^\beta \rho_\theta(I \cap \complement\Lambda_M) d\theta > 0.$$

Proof. The upper and lower bounds given by Lemmas 1 and 2, together with the hypotheses of the theorem imply

$$\int_\alpha^\beta \rho_\theta(I) d\theta > \int_\alpha^\beta \rho_\theta(I \cap \Lambda_M) d\theta.$$

Since

$$\int_\alpha^\beta \rho_\theta(I) d\theta = \int_\alpha^\beta \rho_\theta(I \cap \complement\Lambda_M) d\theta + \int_\alpha^\beta \rho_\theta(I \cap \Lambda_M) d\theta,$$

we have

$$\int_\alpha^\beta \rho_\theta(I \cap \complement\Lambda_M) d\theta > 0.$$ □

The theorem above can be used to analyze the set of boundary conditions θ which give rise to some singular spectrum. For this purpose we only need to consider the case $M = 0$, since Λ_0 happens to be a support for the absolutely continuous part of the spectral measure. Let us consider some examples.

Example 1. If we choose in the theorem above the parameters $k = 1$, $N = 2\pi$ and $M = 0$, then we obtain that the condition

$$\beta - \alpha > \pi|\Lambda_0 \cap I|,$$

where $I = \left(\frac{1}{4}, \frac{9}{4}\right)$, implies

$$\int_\alpha^\beta \rho_\theta \left(I \cap \complement\Lambda_0\right) d\theta > 0.$$

Since $\complement\Lambda_0$ is a support of the singular part we get the existence of a set $B \subset (\alpha, \beta)$ of positive measure such that for $\theta \in B$ the operator H_0 has some singular spectrum in I. Changing the parameters k and N, we can get similar statements for other intervals.

Example 2. The theorem can also be applied to the examples constructed by Remling. He considers in [33] potentials of the form

$$v(x) = \sum_{n=1}^{\infty} g_n v_n(x - a_n),$$

where $g_n > 0, v_n \in L^1[-B_n, B_n], v_n(x) = \chi_{(-B_n,B_n)}(x)w(x)$,

$$w(x) = \int_F \cos(2kx)\, dk.$$

The intervals $[a_n - B_n, a_n + B_n]$ are assumed to be disjoint and the set F, which appears in the definition of the barriers w, is Cantor type in an interval $[a, b]$ with Lebesgue measure any positive number less than $b - a$. The function $\chi_{(-B_n,B_n)}$ denotes the characteristic function of the set $(-B_n, B_n)$.

Let $L_n = a_n - B_n - a_{n-1} - B_{n-1}$ with $a_0 = B_0 = 0$. In [33], under minor assumptions on F the following was proved:

Theorem 6. *Let $g_n = n^{-1/2}, B_n = n^\beta$ with $(2 - 4/\gamma)^{-1} < \beta < \gamma/8$, where $\gamma > 6$, and assume $n^{\beta/2\gamma} L_{n-1}/L_n \to 0$ as $n \to \infty$. Then the half-line Schrödinger operators H_α with potential v given as above satisfy $\sigma_{ac}(H_\alpha) = \sigma_{ess}(H_\alpha) = [0, \infty), \sigma_d(H_\alpha) \cap (0, \infty) = \varnothing$, and $\sigma_{sc}(H_\alpha) \cap (0, \infty) \neq \varnothing$ for a set of boundary conditions α of positive measure.*

In proving this result it is shown that the singular part of the spectral measures ρ_α corresponding to H_α are supported on $F^2 = \{k^2 : k \in F\}$. In this theorem the potential can be chosen to be zero in an interval $[0, N]$.

We can apply Theorem 5 as we did in Example 1 taking $k = 1, N = 2\pi$ and $M = 0$. The requirement to have singular continuous spectrum for a set of boundary conditions of positive measure in (α, β) is

$$\beta - \alpha > \pi|\Lambda_0 \cap I| = |(\complement F^2) \cap I|,$$

where $I = \left(\frac{1}{4}, \frac{9}{4}\right)$. Observe that we can control the measure of F^2 and therefore, by modifying $|F|$, we can choose the length of (α, β).

5. Coexistence for all boundary conditions

As mentioned above, it is possible to have absolutely continuous spectrum and singular spectrum even for *all* boundary conditions. In fact very simple spectral measures generate this coexistence [7]. Consider, for example, a fixed interval I and take a set $E \subset I$ such that for every subinterval $J \subset I$ we have

$$0 < |E \cap J| < |J|,$$

where $|\cdot|$ denotes the Lebesgue measure, that is, E and $\complement E$ are essentially dense in I. We need this property of E in order to get genuinely mixed spectra.

Let

$$u(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

and define $d\mu_{\pi/2} = u\,dx$. We have to define $\mu_{\pi/2}$ outside I so that

$$\int_{\mathbb{R}} \frac{d\mu_{\pi/2}(t)}{1+t^2} < \infty$$

and the necessary decay conditions required by the Gelfand-Levitan inverse theorem [31, Chapter VIII] are satisfied. The measure $\mu_{\pi/2}$ will be the spectral measure of a Sturm-Liouville operator $H_{\pi/2}$ and we denote by μ_β the spectral measures of H_β, where μ_β^s and μ_β^{ac} stand for their singular and absolutely continuous components respectively.

The family of measures μ_β so generated have the following properties:

Theorem 7.

a) $\mu_\beta^s(J) > 0$ *for every subinterval* $J \subset I,\ \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

b) $\mu_\beta^{ac}(J) > 0$ *for every subinterval* $J \subset I,\ \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Before we prove this theorem we need a preliminary result. For real θ consider the family of functions

$$f_\theta(z) = \frac{\cos\theta + z\sin\theta}{\sin\theta - z\cos\theta}\,. \tag{8}$$

Observe that using (5) we have $f_\theta(m_{\pi/2}(z)) = m_\theta(z)$. For each θ, f_θ is an analytic function that maps the upper half-plane into itself. We shall refer to such functions as Pick functions (also known as Nevanlinna or Herglotz functions). Given a Pick function F it has an integral representation of the form

$$F(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t). \tag{9}$$

The integral on the right-hand side is the Cauchy integral of μ in the upper half-plane. The Weyl functions m_α are Pick functions and the spectral measure ρ_α is the measure that appears in the integral representation of these functions.

Let L be a Pick function such that $0 \le \operatorname{Im} L(z) \le \pi$. For any $\alpha \in \mathbb{R}$, set

$$M_\alpha(z) := f_\alpha\big(L(z)\big) \quad \text{and} \quad N_\alpha(z) := f_\alpha\big(\exp L(z)\big).$$

Both M_α and N_α are Pick functions admitting representations similar to (9). We denote by μ_α and ν_α the associated measures that appear in the integral representations. The singular parts of these measures ν_α^s and μ_α^s satisfy the following relation:

Lemma 3. *Define a function* $\alpha : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (0, \frac{\pi}{2})$ *by*

$$\alpha(\beta) = \arctan(\exp\tan\beta) \quad \textit{with } \beta \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

Then

$$\mu_\beta^s = \alpha'(\beta)\nu_{\alpha(\beta)}^s.$$

Sketch of proof. First we show that

$$\lim_{y\downarrow 0}\frac{\operatorname{Im} M_\beta(x+iy)}{\operatorname{Im} N_\alpha(x+iy)}=\alpha'(\beta)\quad \text{for } \mu^s_\beta\text{-a.e. } x. \tag{10}$$

To see this, using the definition of M_β and N_α we have

$$\lim_{y\downarrow 0}\frac{\operatorname{Im}M_\beta(z)}{\operatorname{Im}N_\alpha(z)}=\lim_{y\downarrow 0}\frac{\operatorname{Im}L(z)}{\operatorname{Im}\exp L(z)}\left|\frac{\sin\alpha-\exp L(z)\cos\alpha}{\sin\beta-L(z)\cos\beta}\right|^2.$$

From the definition of α

$$\frac{\sin\alpha-\exp L(z)\cos\alpha}{\sin\beta-L(z)\cos\beta}=\frac{\cos\alpha}{\cos\beta}\cdot\frac{\exp(\tan\beta)-\exp L(z)}{\tan\beta-L(z)}. \tag{11}$$

It is well known that for μ^s_β-a.e. x the Cauchy integral of μ_β at $x+i\varepsilon$ tends to infinity as $\varepsilon\to 0$. Therefore $M_\beta(x+i\varepsilon)\longrightarrow\infty$ as $\varepsilon\downarrow 0$. The formula for f_β and the definition of M_β now imply that for μ^s_β-a.e. x, $L(x+i\varepsilon)\xrightarrow{\varepsilon\downarrow 0}\tan\beta$. Hence the expression in the right-hand side of (11) tends to

$$\frac{\cos\alpha}{\cos\beta}\exp(\tan\beta)\qquad\text{when } \varepsilon\downarrow 0$$

and we obtain

$$\left|\frac{\sin\alpha-\exp L(z)\cos\alpha}{\sin\beta-L(z)\cos\beta}\right|^2\xrightarrow{\varepsilon\downarrow 0}\left(\frac{\cos\alpha}{\cos\beta}\right)^2(\exp\tan\beta)^2\quad\text{for } \mu^s_\beta\text{-a.e. } x. \tag{12}$$

Once (10) is obtained, it follows that

$$\left|\frac{\operatorname{Im} M_\beta(x+iy)}{\operatorname{Im} N_\alpha(x+iy)}-\frac{\mu_\beta(x-y,\,x+y)}{\nu_\alpha(x-y,\,x+y)}\right|\xrightarrow{y\downarrow 0}0$$

for μ^s_β-a.e. x and from here the assertion of the lemma follows. □

Proof of Theorem 7. Let

$$F_{\nu_{\pi/2}}(z):=\exp\left(C+\int_{\mathbb{R}}\left(\frac{1}{\lambda-z}-\frac{\lambda}{\lambda^2+1}\right)d\mu_{\pi/2}(\lambda)\right), \tag{13}$$

where C is chosen so that

$$C+\int_{\mathbb{R}}\left(\frac{1}{\lambda-z}-\frac{\lambda}{\lambda^2+1}\right)d\mu_{\pi/2}(\lambda)=m_{\pi/2}(z),$$

that is, $\log F_{\nu_{\pi/2}}$ is the Weyl function of some Sturm-Liouville operator with boundary condition $\alpha=\frac{\pi}{2}$. $F_{\nu_{\pi/2}}$ is a Pick function. Then recalling the definition of $\mu_{\pi/2}$ we get

$$u(x)=\frac{1}{\pi}\arg F_{\nu_{\pi/2}}(x+i0),$$

where arg stands for the principal branch of argument, taking values in $(-\pi,\pi]$. Therefore

$$\operatorname{Im} F_{\nu_{\pi/2}}(x+i0)=0 \text{ for a.e. } x\in I.$$

Since the support of the absolutely continuous part of ν_α is the set

$$\{x \mid \operatorname{Im} F_{\nu_{\pi/2}}(x+i0) > 0\},$$

it follows that ν_α is purely singular in I for every $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Given an interval $J \subset I$ assume that $\nu_{\pi/2}(J) = 0$. Then $F_{\nu_{\pi/2}}(z)$ can be extended analytically across J and from (13) the same follows for $m_{\pi/2}(z)$. Since this implies $\mu_{\pi/2}(J) = 0$, we get a contradiction to the construction of $\mu_{\pi/2}$. Hence $\nu_{\pi/2}(J) > 0$ for every $J \subset I$. Therefore $\nu^s_\alpha(J) > 0$ for every $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (see for instance [14]).

Now to obtain part a) in the theorem we just recall Lemma 3, and we have for every Borel set A and every $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\mu^s_\beta(A) = \alpha'(\beta)\nu^s_{\alpha(\beta)}(A).$$

Part b) follows from the well-known stability of the absolutely continuous part.

□

6. Some inverse spectral theory

Let us now refer briefly to the inverse spectral theory of regular Sturm-Liouville operators in relation to dependence on the boundary conditions. We shall consider the operator $-d^2/dx^2 + q$ in $L^2(0,1)$ with boundary conditions

$$\begin{aligned} y(0)\cos\alpha - y'(0)\sin\alpha = 0, \\ y(1)\cos\theta - y'(1)\sin\theta = 0, \end{aligned} \tag{14}$$

where $q \in L^1(0,1)$. We denote this operator again by H_α (we shall mostly consider various values of α for fixed θ). It was G. Borg in 1946 [2] who proved the following:

Theorem 8. *The spectra of H_α for two values of α uniquely determine q.*

This statement should be interpreted as follows. Denote by $\sigma(q;\alpha)$ the spectrum of $-d^2/dx^2 + q$, with boundary conditions (14), where $\alpha \in [0,\pi)$. Let $q_1, q_2 \in L^1(0,1)$ be such that $\sigma(q_1;\alpha_1) = \sigma(q_2;\alpha_1)$ and $\sigma(q_1;\alpha_2) = \sigma(q_2;\alpha_2)$ for some α_1, α_2 ($\alpha_1 \neq \alpha_2$). Then $q_1(x) = q_2(x)$ for a.e. $x \in [0,1]$.

If we know more about the potential, then we need less information about the spectra as the following theorem of Hochstadt-Lieberman [21] states:

Theorem 9. *The spectrum of H_α for one value of α together with the values of q on $[0,\frac{1}{2}]$ uniquely determine q on $[\frac{1}{2},1]$.*

For interesting generalizations of these theorems to the case of matrix equations, see [28] as well as the contribution by M. Malamud [29] in this volume.

It is remarkable that we need exactly half of the potential q, since the knowledge of q on $[0,\frac{1}{2}-\varepsilon]$ for any $\varepsilon > 0$ and the spectrum for one boundary condition is not enough to reconstruct q. This can be seen for instance for Dirichlet boundary conditions if we observe that in that case reflecting q at the point $\frac{1}{2}$ does not change the spectrum and then take

$$q(x) = \begin{cases} c & \text{if } x \in [0, \frac{1}{2} - \varepsilon], \\ v(x) \neq c & \text{if } x \in (\frac{1}{2} - \varepsilon, \frac{1}{2}), \\ c & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

so that q and its reflection will coincide in $[0, \frac{1}{2} - \varepsilon]$ but will differ on $(\frac{1}{2} - \varepsilon, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1}{2} + \varepsilon)$. In [15] the following was proved (we refer to that reference for a precise definition of the meaning of "half the spectrum"; for example it suffices to enumerate the eigenvalues in increasing order and take the lowest two eigenvalues and then every second one):

Theorem 10. *Half the spectrum of one H_α together with the values of q on $[0, \frac{3}{4}]$ uniquely determine q.*

In [8] results of the following kind are presented.

Theorem 11. *Let $\alpha_1, \alpha_2, \alpha_3 \in [0, \pi)$ and denote by σ_j the spectrum of $H_{\alpha_j}, j = 1, 2, 3$. Assume $S_j \subset \sigma_j$, and suppose that for all sufficiently large $\lambda_0 > 0$ we have*

$$\begin{aligned} &\#\{\lambda \in \{S_1 \cup S_2 \cup S_3\} \text{ with } \lambda \leq \lambda_0\} \\ &\qquad \geq \frac{2}{3}\#\{\lambda \in \{\sigma_1 \cup \sigma_2 \cup \sigma_3\} \text{ with } \lambda \leq \lambda_0\} - 1. \end{aligned}$$

Then q is uniquely determined a.e. on $[0, 1]$.

In particular, two thirds of three spectra determine q. Another result (involving the values $\alpha = \pi/2$ and $\alpha = 0$) is the following:

Theorem 12. *Let σ_N and σ_D be the eigenvalues of $H_{\pi/2}$ and H_0 respectively. Let $S_N \subset \sigma_N, S_D \subset \sigma_D$. Fix $a \in (0, 1)$. Suppose that for all $\lambda_0 > 0$ sufficiently large we have*

$$\begin{aligned} &\#\{\lambda \in \{S_N \cup S_D\} \text{ with } \lambda \leq \lambda_0\} \\ &\qquad \geq (1 - a)\#\{\lambda \in \{\sigma_N \cup \sigma_D\} \text{ with } \lambda \leq \lambda_0\}. \end{aligned}$$

Then S_N, S_D and q on $[0, a]$ uniquely determine q a.e. on $[0, 1]$.

For example if $a = \frac{1}{4}$, then knowing q on $[0, \frac{1}{4}]$, all the Neumann eigenvalues and half the Dirichlet eigenvalues, uniquely determine q a.e. on $[0, 1]$. Generalizations of some of the above results can be found in a paper by M. Horváth [22].

The strategy to prove the theorems just mentioned is to use the fundamental result of Marchenko, see Theorem 13 below, and to prove a general theorem that knowing m at points $\lambda_0, \lambda_1, \ldots$ determines m as long as $\{\lambda_n\}$ has enough density.

Since we are considering the regular case, we define the Weyl function m as

$$m_\theta(z) = \frac{u_\theta'(0, z)}{u_\theta(0, z)}, \quad z \in \mathbb{C},$$

where $u_\theta(x,z)$ solves $-u''(x,z)+q(x)u(x,z)=zu(x,z)$ with boundary condition (14) at $x=1$.

Theorem 13. [30] *The Weyl m-function uniquely determines q a.e. in $[0,1]$.*

Another result that should be possible to prove for Sturm-Liouville operators (I know the proof only for Dirac systems [9]) is the following statement in which, for $q \in L^2(0,1)$ and $\theta \in [0,\pi)$, we denote by $\mu_m(q;\theta)$ $(m=1,2,3,\dots)$ the eigenvalues of $H=-d^2/dx^2+q$ in $L^2(0,1)$ taken in increasing order, with boundary conditions (14) in the case $\alpha=0$:

Let $a \in [0,1]$ and let l,k be positive integers satisfying $\frac{1}{l}+\frac{1}{k} \geq 2a$. For some $q_1,q_2 \in L^2(0,1)$ and $\theta_1,\theta_2 \in [0,\pi)$ with $\theta_1 \neq \theta_2$, suppose that $\mu_{ln}(q_1;\theta_1)=\mu_{ln}(q_2;\theta_1)$ and $\mu_{kn}(q_1;\theta_2)=\mu_{kn}(q_2;\theta_2)$ for each $n \in \mathbb{N}$. Suppose moreover that $q_1(x)=q_2(x)$ a.e. on the interval $[a,1]$. Then $q_1(x)=q_2(x)$ a.e. in $[0,1]$.

Thus the sets $\{\mu_{ln}(q;\theta_1) \mid n \in \mathbb{N}\}$ and $\{\mu_{kn}(q;\theta_2) \mid n \in \mathbb{N}\}$ together with $q|_{[a,1]}$ uniquely determine q. The result should also hold if we allow $l=\infty$ or $k=\infty$, where for example in the case $l=\infty$, the set $\{\mu_{kn}(q;\theta) \mid n \in \mathbb{N}\}$ for some θ, together with $q|_{[a,1]}$, uniquely determine q a.e. in $[0,1]$.

For particular values of a,k,l, one would then obtain:

- Borg type theorem: two spectra uniquely determine q on $[0,1]$ ($a=1$, $l=k=1$).
- Hochstadt-Lieberman type theorem: one spectrum and q on $[1/2,1]$ uniquely determine q on $[0,1]$ ($a=1/2$, $l=1$, $k=\infty$).

Actually the above result would include many more general cases such as, for example:

- half of one spectrum and q on $[1/4,1]$ uniquely determine q on $[0,1]$ ($a=1/4$, $l=2$, $k=\infty$).
- half of two spectra and q on $[1/2,1]$ uniquely determine q on $[0,1]$ ($a=1/2$, $l=k=2$).

In results of the above kind involving spectra with different boundary conditions at one endpoint, it is important to take note of the endpoint at which the boundary condition may vary, relative to that at which the potential is given. As an example, consider the following situation: suppose that $q_1(x)=q_2(x)$ a.e. on $[0,1/2]$ and that $\mu_n(q_1;\theta_1)=\mu_n(q_2;\theta_2)$ for all $n \in \mathbb{N}$ and some $\theta_1,\theta_2 \in [0,\pi)$. A result of Hald [18] implies that $q_1(x)=q_2(x)$ a.e. on $[0,1]$ (and hence $\theta_1=\theta_2$). A similar uniqueness result would not, however, hold if the boundary condition at $x=0$, rather than at $x=1$, was varied. This may be shown as follows (see [5]).

Let p be an arbitrary but fixed point in the open interval (0,1). For $x \in [p,1]$ consider the problem

$$-y''(x)+v(x)y(x)=\mu y(x) \tag{15}$$

with boundary conditions

$$\begin{aligned} y(p) - y'(p) &= 0, \\ y(1) &= 0, \end{aligned}$$

where v is an arbitrary integrable function. We denote by $\{\mu_i\}_{i=0}^{\infty}$ the eigenvalues and by $\{\phi_i\}_{i=0}^{\infty}$ the corresponding eigenfunctions.

Now consider the potential q defined for $x \in [0,1]$ as follows:

$$q(x) = \begin{cases} 1 + \mu_0 & \text{if } x \in [0,p] \\ v(x) & \text{if } x \in (p,1] \end{cases}$$

and the problem

$$-y''(x) + q(x)y(x) = \lambda y(x), \qquad x \in [0,1] \tag{16}$$

with boundary conditions

$$\begin{aligned} y(0) - y'(0) &= 0, \\ y(1) &= 0. \end{aligned} \tag{17}$$

Let us define

$$g_0(x) = \begin{cases} \sqrt{2}\, e^x & \text{for } x \in [0,p], \\ \phi_0(x) & \text{for } x \in (p,1], \end{cases}$$

where we have normalized ϕ_0 in such a way that $\phi_0(p) = \sqrt{2}\, e^p$ holds. The function g_0 is continuously differentiable, solves (16), (17) for $\lambda = \mu_0$, and since μ_0 is the first eigenvalue of (15), g_0 does not have any zeros in $[0,1)$. Define

$$q_1(x) := q(x) - 2\frac{d^2}{dx^2} \log\left(1 + \int_0^x g_0^2(s)ds\right).$$

It is easy to see that q_1 is integrable on $[0,1]$. The following result is proved in [5]:

Theorem 14. *The problems*

$$\begin{aligned} -y'' + q_1(x)y &= \lambda y \qquad x \in [0,1], \\ y(0) + y'(0) &= 0, \\ y(1) &= 0, \end{aligned}$$

and

$$\begin{aligned} -y'' + q(x)y &= \lambda y \qquad x \in [0,1], \\ y(0) - y'(0) &= 0, \\ y(1) &= 0, \end{aligned}$$

have the same spectrum $\{\lambda_i\}_{i=1}^{\infty}$. *In addition to this:* $q_1(x) = q(x)$ *for* $x \in [0,p]$ *and* $q_1 \neq q$ *as elements of* $L^1(0,1)$.

Acknowledgements

I thank Professors Werner Amrein, Andreas Hinz, David Pearson and the other organizers for a very interesting Conference on the occasion of the 200th anniversary of the birth of Charles François Sturm, for suggesting to me to write this paper and for all the work and patience in the careful revision of this contribution. I would like to thank Prof. Hubert Kalf for providing me with valuable information regarding the study of singular continuous spectra, Olga Tchebotareva for pertinent comments and Francine Gennai–Nicole for her help and kindness during my stay in Geneva.

References

[1] N. Aronszajn, On a problem of Weyl in the theory of singular Sturm–Liouville equations, Amer. J. Math. **79** (1957), 597–610.

[2] G. Borg, Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe, Acta Math. **78** (1946), 1–96.

[3] E.A. Coddington and N. Levinson, On the nature of the spectrum of singular second-order linear differential equations, Canad. J. Math. **3** (1951), 335–338.

[4] P. Deift and R. Killip, On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials, Comm. Math. Phys. **203** (1999), 341–347.

[5] R. del Rio, On Boundary Conditions of An Inverse Sturm–Liouville Problem, SIAM Journal on Applied Mathematics **50** (1990), 1745–1751.

[6] R. del Rio, On a problem of P. Hartman and A. Wintner, Memorias del XXVII Congreso Nacional de la Sociedad Matemática Mexicana, Aportaciones Matemáticas, Serie Comunicaciones **16** (1995), 119–123.

[7] R. del Rio, S. Fuentes and A. Poltoratski, Families of Spectral Measures with mixed types, Oper. Theory Adv. Appl. **132** (2002), 131–140.

[8] R. del Rio, F. Gesztesy and B. Simon, Inverse Spectral Analysis with Partial Information on the Potential III. Updating boundary conditions, International Mathematical Research Notes **15** (1997), 751–758.

[9] R. del Rio and B. Grébert, Inverse Spectral Results for AKNS Systems with Partial Information on the Potentials, Mathematical Physics, Analysis and Geometry **4** (2001), 229–244.

[10] R. del Rio, N. Makarov and B. Simon, Operators with Singular Continuous Spectrum II: Rank One Operators, Comm. Math. Phys. **165** (1994), 59–67.

[11] R. del Rio and A. Poltoratski, Spectral Measures and Category, Oper. Theory Adv. Appl. **108** (1999), 149–159.

[12] R. del Rio, B. Simon and G. Stolz, Stability of Spectral Types of Sturm-Liouville Operators, Math. Res. Let. **1** (1994), 437–450.

[13] R. del Rio and O. Tchebotareva, Boundary conditions of Sturm-Liouville operators with mixed spectra, Math. Anal. Appl. **288** (2003), 518–529.

[14] M.S.P. Eastham and H. Kalf, *Schrödinger-type operators with continuous spectra*, Pitman Advanced Publishing Program, Boston, 1982.

[15] F. Gesztesy and B. Simon, Inverse spectral analysis with partial information on the potential, II. The case of discrete spectrum, Trans. Amer. Math. Soc. **352** (2000), 2765–2787.

[16] D. Gilbert, Asymptotic Methods in the Spectral Analysis of Sturm-Liouville Operators, in this volume.

[17] D. Gilbert and D. Pearson, On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators, J. Math. Anal. Appl. **128** (1987), 30–56.

[18] O.H. Hald, Inverse eigenvalue problems for the mantle, Geophys. J. Royal Astron. Soc. **62** (1980), 41–48.

[19] P. Hartman and A. Wintner, An oscillation theorem for continuous spectra, Proc. Nat. Acad. Sci. USA **33** (1947), 376–379.

[20] P. Hartman and A. Wintner, A separation theorem for continuous spectra, Amer. J. Math. **71** (1949), 650–662.

[21] H. Hochstadt and B. Lieberman, An inverse Sturm-Liouville problem with mixed given data, SIAM J. Appl. Math. **34** (1978), 676–680.

[22] M. Horváth, On the inverse spectral theory of Schrödinger and Dirac operators, Trans. Amer. Math. Soc. **353** (2001), 4155–4171.

[23] T. Kato, Perturbation of continuous spectra by trace class operators, Proc. Japan Acad. **33** (1957), 260–264.

[24] W. Kirsch, S. Molchanov and L. Pastur, One-dimensional Schrödinger operators with high potential barriers, Oper. Theory Adv. Appl. **57** (1992), 163–170.

[25] A. Kiselev, Y. Last and B. Simon, Stability of singular spectral types under decaying perturbations, J. Funct. Anal. **198** (2003), 1–27.

[26] Y. Last, Spectral Theory of Sturm-Liouville Operators on Infinite Intervals: A Review of Recent Developments, in this volume.

[27] N. Makarov, Personal communication.

[28] M. Malamud, Uniqueness questions in inverse problems for systems of differential equations on a finite interval, Trans. Moscow Math. Soc. **60** (1999), 173–224.

[29] M. Malamud, Uniqueness of the Matrix Sturm-Liouville Equation given a Part of the Monodromy Matrix, and Borg Type Results, in this volume.

[30] V.A. Marchenko, Some questions in the theory of one-dimensional linear differential operators of the second order, I, Trudy Moskov. Mat. Obšč. **1** (1952), 327–420 (Russian); Amer. Math. Soc. Transl. **101** (1973), 1–104.

[31] M. Naimark, *Linear Differential Operators II*, Frederick Ungar Publishing Co., New York, 1968.

[32] C. Remling, Essential spectrum and L_2-solutions of one-dimensional Schrödinger operators, Proceedings of the Amer. Math. Soc. **124** (1996), 2097–2100.

[33] C. Remling, Embedded singular continuous spectrum for one-dimensional Schrödinger operators, Trans. Amer. Math. Soc. **351** (1999), 2479–2497.

[34] C. Remling, Universal bounds on spectral measures of one-dimensional Schrödinger operators, J. Reine Angew. Math **564** (2003), 105–117.

[35] M. Rosenblum, Perturbation of the continuous spectrum and unitary equivalence, Pacific J. Math. **7** (1957), 997–1010.

[36] B. Simon, Spectral analysis of rank one perturbations and applications, CMR Proc. Lecture Notes **8** (1995), 109–149.

[37] G. Stolz, Bounded solutions and absolute continuity of Sturm-Liouville operators, J. Math. Anal. Appl. **169** (1992), 210–228.

[38] C. Sturm, Mémoire sur les Équations différentielles linéaires du second ordre, J. Math. Pures Appl. **1** (1836), 106–186.

[39] C. Sturm, Mémoire sur une classe d'Équations à différences partielles, J. Math. Pures Appl. **1** (1836), 373–444.

[40] O. Tchebotareva, *Schrödinger Operators and Spectral Types*, PhD Thesis, UNAM, México, 2004.

[41] H. Weyl, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen, Math. Ann. **68** (1910), 220–269.

[42] H. Weyl, Über beschränkte quadratische Formen, deren Differenz vollstetig ist, Rend. Circ. Mat. Palermo **27** (1909), 373–392.

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