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COEXISTENCE OF SPECTRA IN RANK-ONE PERTURBATION PROBLEMS

R. DEL RIO, S. FUENTES AND A. POLTORATSKI

ABSTRACT. We study the behavior of spectral functions corresponding to selfadjoint operators of the form $A + \lambda\langle\varphi, \cdot\rangle\varphi$. The focus is on the coexistence of absolutely continuous and singular spectra for values of the real parameter λ in a given set B. For almost all points of B it is possible to construct a family of rank one perturbations with mixed spectra.

1. Introduction

In this note we shall study the behavior of different parts of the spectrum of an operator under small perturbations. Such problems became an object of active research in recent years mostly due to the applications to differential equations and mathematical physics. In such applications the operators are usually unitary or self-adjoint and acting in a separable Hilbert space.

Let A_0 be a cyclic self-adjoint operator, φ its cyclic vector and μ the corresponding spectral measure. Denote by A_λ the rank one perturbations of A_0:

$$A_\lambda = A_0 + \lambda(\cdot, \varphi)\varphi, \quad \lambda \in \mathbb{R}. \tag{1.1}$$

In case φ is not in the Hilbert space where A_0 acts, this expression has to be understood formally. For example, φ could be a generalized vector in the space $\mathcal{H}_{-2}$ from the scale of Hilbert spaces associated to A_0, see [1]. We discuss this further in the remark after the proof of Lemma (2.9).

Let μ_λ be the spectral measures of A_λ corresponding to φ. The general problem related to such a family is to understand how the measures μ_λ change when one varies the parameter (the coupling constant) λ.

Such problems appear in many applications. For instance, if one considers a discrete Schrödinger operator and starts changing the potential at one of the points of the lattice one will obtain a family of self-adjoint rank-one perturbations. Our question then translates into the problem of predicting the changes in the dynamical properties of the quantum system caused by small perturbations in the potential field. Similarly, if one takes a Sturm-Liouville differential operator on the half-axis and begins varying the boundary condition at 0 one again obtains an example of a family A_λ. The same problem appears in many other areas including self-adjoint extensions of symmetric operators and spaces of pseudocontinuable functions in the unit disk. For more on these connections see [9], [17] and [14].

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Every measure μ_λ can be in a standard way decomposed into the sum of its absolutely continuous and singular parts. In quantum dynamics the spectrum of an operator represents the set of all admissible energies for the underlying quantum system. The spectral subspaces corresponding to the different parts of the spectral measure consist of vectors (states) displaying different dynamic behavior. It is, therefore, important to understand the interplay between different parts of μ_λ under small perturbations. In this paper we focus on the correlations between the absolutely continuous and singular spectrum.

By the Kato-Rosenblum theorem the absolutely continuous spectrum is invariant under such small perturbations. At the same time, examples obtained in recent years show that the singular spectrum is extremely unstable under the change of the parameter λ in particular in "mixed" situations when both singular and absolutely continuous spectra are present. Such "mixed" measures correspond to the systems admitting states with different dynamic properties, which seems to be an especially interesting case in many situations. The usual place for the singular spectrum of A_λ is the gaps of the absolutely continuous spectrum. With the change of λ the isolated eigenvalues of A_λ move inside their respective gaps and often dissappear when they hit the edge of the gap. Looking at such simple examples one may get an impression that in the case when there are no gaps in the a.c. spectrum, the dense singular spectrum, even if it originally existed, must be immediately destroyed when the parameter λ starts changing. In fact, most of the existing examples seem to support this intuitive model (see for instance the papers by Naboko [12] and Simon [18] where the singular spectrum exists only for a set of parameters of Lebesgue measure zero). However, the example obtained in [3] produced a family of operators with dense absolutely continuous and dense singular spectrum for all values of the parameter.

In many applications the perturbations occur randomly, i. e. the parameter λ is a random variable with certain distribution on the real line. If the problem is considered from such a point of view, one can say that the probability of coexistence of dense singular and absolutely continuous spectra in systems constructed in [12] and [18] is 0 (for any a. c. distribution of λ) and 1 for the example from [3].

The next natural question seems to be if the coexistence is possible for the sets of parameters other than the whole line or zero-sets. In the next section, with an addition of new analytic tools, we are able to complete the construction of [3] to obtain the following result.

THEOREM (1.2). *For any measurable set $B \subset \mathbb{R}$ there exists a family of rank-one perturbations $\{A_\lambda\}_{\lambda\in\mathbb{R}}$ such that A_λ has dense absolutely continuous and dense singular spectrum for almost every $\lambda \in B$ and dense absolutely continuous (but no singular) spectrum for almost every $\lambda \notin B$.*

Although in probabilistic settings the sets of measure zero are negligible, it would be interesting to know if the words "almost every" could be removed from the last statement. Such a problem requires a completely different approach. As an example, in this paper we show that there exists a family of operators such that the spectrum is dense absolutely continuous for *all* $\lambda \in [0, 1]$ and dense absolutely continuous plus dense singular for *all* $\lambda \in \mathbb{R}\backslash[0,1]$. In this construction we utilize the Krein spectral shift function. While a similar technique may provide the answer for other simple sets B, the general question remains open.

2. The main proof

Every Pick function $F(z)$ (an analytic function which takes the upper half-plane $\mathbb{C}_+$ into itself) has an integral representation of the form

$$F(z) = a + bz + K\mu \tag{2.1}$$

where $a, b \in \mathbb{R}$, $b \geq 0$, μ is a non negative Borel measure which satisfies

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$$

and $K\mu$ is its Cauchy integral in the upper half-plane:

$$K\mu = \int_{\mathbb{R}} \left[\frac{1}{t-z} - \frac{t}{1+t^2}\right] d\mu(t)\,. \tag{2.2}$$

The term bz can be viewed as the Cauchy integral of a point mass b at infinity. Conversely, any function of this form is an analytic map from the upper half-plane into itself, see for instance [16]. We will also denote by $P\mu$ the Poisson integral of μ:

$$P\mu(x+iy) = Im K\mu(x+iy) = \int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2} d\mu(t)\,.$$

We will use the notation $C\gamma$ for the Cauchy integral of the measure γ with a different kernel:

$$C\gamma(z) = \int_{\mathbb{R}} \frac{d\gamma(x)}{x-z}\,. \tag{2.3}$$

Such a transform can only be applied if the measure satisfies stronger restriction

$$K\mu = \int_{\mathbb{R}} \left[\frac{1}{t-z} - \frac{t}{1+|t|}\right] d\mu(t)\,. \tag{2.4}$$

Proof of Theorem (1.2).

Part I.

Let $B \subset \mathbb{R}$ be a measurable set. Here we will assume that B is bounded. Such a restriction is not important, but will shorten the proof. In this part we construct a family A_λ such that the spectrum is absolutely continuous for a. e. $\lambda \in \bar{B} = \mathbb{R} \setminus B$ and absolutely continuous plus singular for a. e. $\lambda \in B$. In the second part we will make all parts of the spectrum dense in $\mathbb{R}$. For any measurable set S we denote by $|S|$ the measure of S with respect to the normalized Lebesgue measure $dx/(1+x^2)$.

Without loss of generality (abbreviated WLOG henceforth) $0 \notin B$. Denote $E = -1/B = \{x| -1/x \in B\}$. Then there exist open sets E_n containing E such that $E_1 \supset E_2 \supset \ldots$ and $|E_k \setminus E| \to 0$.

If $O \subset \mathbb{R}$ is an open set and ϵ is a positive number, we denote by $\Omega(O, \epsilon)$ the open domain in the upper half-plane constructed in the following way. Let $O = \cup I_n$ where I_n are disjoint open intervals. Denote by Q_n the open rectangle in the upper half-plane with base I_n and of height greater than ϵ. We will denote

$$\Omega(O, \epsilon) = (\cup Q_n) \cup \{z | \operatorname{Im} z > \epsilon\}.$$

Let $\{\epsilon_n\}$ be a decreasing sequence of positive numbers tending to 0 (to be chosen later). For each n consider the domain $\Omega(E_n, \epsilon_n)$, see fig. 1.

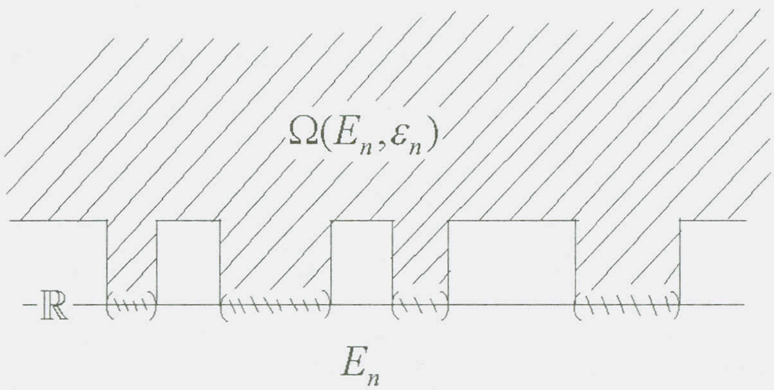

Fig. 1

Put

$$\Gamma = \left(\cap_n \Omega(E_n, \epsilon_n)\right) \cup \{z / \operatorname{Im} z > \exp(-1/|\operatorname{Re} z|)\},$$

see fig. 2.

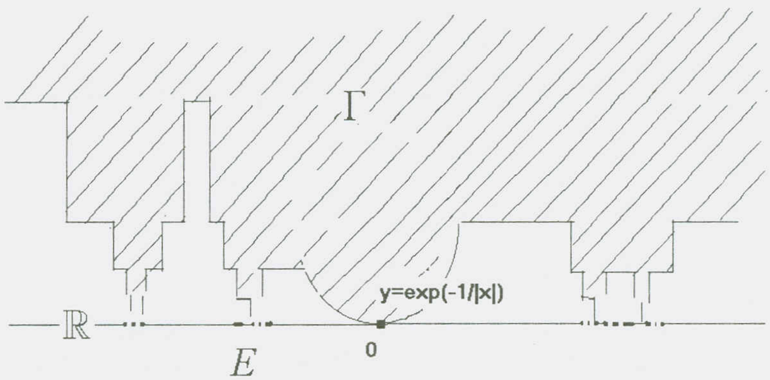

Fig. 2

It can be easily verified that Γ is an open domain. Hence by the Riemann Theorem there exists a conformal map ϕ from the upper half-plane to Γ. Since ϕ maps $\mathbb{C}_+$ into itself, it is a Pick function allowing the representation (2.1). Moreover, ϕ can be chosen so that $a, b = 0$ by requiring $\phi(\infty) = 0$. We denote the measure on $\mathbb{R}$ appearing in the representation (2) for ϕ by μ_0. As we will demonstrate shortly, μ_0 is finite (this is the reason we had to add the region $\{z/\operatorname{Im} z > \exp(-1/|\operatorname{Re} z|)\}$ to Γ), see remark at the end of the proof. Therefore the vector $\varphi = 1$ is in $L^2(\mu_0)$ and we can consider a self-adjoint cyclic operator A_0 of multiplication by x in $L^2(\mu_0)$ and put $\varphi = 1$. Then μ_0 is the spectral measure of A_0 corresponding to φ. We denote by A_λ the corresponding family of rank-one perturbations and by μ_λ the spectral measures of these operators. We claim that the family A_λ posesses the desired properties (after the proper choice of ϵ_n).

Since B is bounded, E does not contain a neighborhood of 0. Hence Γ has a Dini-smooth corner of opening π at ∞ (see [15] for the definition). Since $\phi(\infty) = 0$, by Theorem 3.9 [15] ϕ can be extended continuously up to the boundary in a neighborhood of ∞ and $|\operatorname{Re}\phi(x)| > C\frac{1}{|x|}$ for $x \in \mathbb{R}$ if $|x|$ is large enough. The continuity implies that the singular part of μ_0 is supported on a bounded set. At the same time, for x far enough from the origin the density f of the absolutely continuous part of μ_0 can be estimated as follows:

$$f(x) = \operatorname{Im}\phi(x) = \exp(-1/|\operatorname{Re}\phi(x)|) < \exp(-C|x|)\,.$$

Hence μ_0 is finite and $\varphi = 1 \in L^2(\mu_0)$. Also, since

$$\lim_{|x|\to\infty} \operatorname{Im}\phi(x) = 0\,,$$

and $\phi = K\mu_0$ we must have

$$\int_{\mathbb{R}} \frac{t}{1+t^2}\,dt = 0$$

and $\phi = K\mu_0 = C\mu_0$.

After [15], we say that a point on the boundary of a complex domain is sectorially accessible if there exists an open triangle with a vertex at this point that lies completely within the domain. If f is a conformal map from $\mathbb{C}_+$ to W we denote by *Sect*(f) the set of points x on $\mathbb{R}$ at which f has nontangential limit $f(x) \in \partial W$ that is sectorially accessible in W.

Now we want to show that if ϵ_n are chosen in a right way then almost all points from E are sectorially accessible in Γ.

Let O be an open set $O = \cup I_n$ for some disjoint open intervals I_n. If $|I_n| > 2\epsilon$ we denote by I_n^ϵ the open interval with the same center of length $|I_n| - 2\epsilon$. If $|I_n| \le 2\epsilon$ then I_n^ϵ stands for the empty set. After that we define $O^\epsilon = \cup I_n^\epsilon$. Note that $|O \setminus O^\epsilon| \to |O|$ as $\epsilon \to 0$.

For each open set E_n choose ϵ_n so that $|E_n \setminus E_n^{\epsilon_n}| < 1/2^n$. It is not difficult to see that if a point x in E is not sectorially accessible from Γ then it must belong to $E_n \setminus E_n^{\epsilon_n}$ for arbitrarily large n's. Hence x must belong to $\cup_{n\ge N} E_n \setminus E_n^{\epsilon_n}$ for every N. But $|\cup_{n\ge N} E_n \setminus E_n^{\epsilon_n}| \le 1/2^{N-1}$. Therefore there is at most a zero set of such unaccessible points x.

To see that all μ_λ have absolutely continuous parts, notice that all points on the boundary of Γ that are not on the real line are sectorially accessible. Therefore, by The McMillan Sector Theorem, see [15], the preimage of such points (the set where ϕ takes such nontangential boundary values) is of positive measure. Hence, there is a set of positive measure where the nontangential boundary values of $C\mu_0$ have positive imaginary parts. This means that μ_0, and therefore all μ_λ, have nontrivial absolutely continuous parts.

Now let us show that μ_λ have non-trivial singular parts for a. e. $\lambda \in B$. The McMillan Twist Theorem [15] implies that ϕ has a nontangential derivative at a. e. point of *Sect*(ϕ). Together with the Sector Theorem this means that, since a. e. point in E is sectorially accessible, for a. e. $y \in E$ there exists $x \in \mathbb{R}$ such that the nontangential limit $\phi(x)$ is equal to y and ϕ has an angular derivative at x. It is well known (see for instance [17]) that this implies that $\mu_{-1/y}$ has a point mass at x. Recall that $E = -1/B$.

To fininsh this part of the proof we will show that μ_λ may have non-trivial singular parts only for a set of measure zero of $\lambda \notin B$. Each open set E_n is a union of disjoint open intervals. Let us denote by S the set which contains the intersection of all open sets E_n and the endpoints of all such open intervals for all sets E_n. Then $S \setminus E$ is a zero set. Since ϕ must tend to $-1/\lambda$ nontangentially $(\mu_\lambda)_s$-a.e., it is enough to show that all the real nontangential limits of ϕ must belong to S. This certainly follows from general facts of the theory of conformal maps, but instead of referring the reader to those results we decided to include the following elementary argument. Let $y \notin S$ and suppose that $\phi(x+ih) \to y$ as $h \to 0$. Suppose that $y \notin E_n = \cup I_k$. There must exist $\delta > 0$ such that $|\phi(x+ih) - y| < \epsilon_n/2$ and $|\phi(x+ih) - y| < \inf_{x \in E_n} \exp(-1/|x|)$ for $h < \delta$. This means that for all $h < \delta$ the point $\phi(x+ih)$ stays inside of the same rectangle Q_k. But since $y \notin E_n$ and it is not an endpoint of the interval I_k (the base of Q_k), $\phi(x+ih)$ can never reach y staying inside Q_k.

Part II.

To make all parts of all μ_λ constructed in Part I dense we can apply Lemma (2.9) below together with the result of [3]. As was shown in [3] there exists a self-adjoint operator T_0 and its cyclic vector ψ such that each measure from the corresponding family of spectral measures ν_λ has absolutely continuous and singular parts, both densely supported in $\mathbb{R}$. Let us define the measure η_0 as

$$\eta_0 = \int_{\mathbb{R}} \nu_\lambda d\mu_0(\lambda).$$

Once again we can choose a self-adjoint operator A_0 and its cyclic vector φ so that η_0 is the corresponding spectral measure. Denote by A_λ the rank-one perturbations defined by (1.1) and by η_λ the spectral measures of A_λ. Then by Lemma (2.9)

$$\eta_\lambda = \int_B \nu_\xi d\mu_\lambda(\xi).$$

In this way all η_λ will have densely supported absolutely continuous parts, densely supported singular parts for a. e. $\lambda \in B$ and no singular part for a. e. $\lambda \notin B$. $\square$

Let us consider the Pick functions

$$f_\lambda(z) = \frac{z}{1+\lambda z}, \quad \lambda \in \mathbb{R}. \tag{2.5}$$

It is well known (see [17]) that if γ is a spectral measure of A_0 corresponding to ϕ and γ_λ are spectral measures of the perturbations A_λ then

$$C\gamma_\lambda(z) = f_\lambda(C\gamma(z)), \ \lambda \in \mathbb{R}.$$

In such cases we will say that $\{\gamma_\lambda\}_{\lambda \in \mathbb{R}}$ is the family generated by γ. Notice that $\gamma_0 = \gamma$

The following result, which is essentially Lemma 5.3 of [4], will be useful

LEMMA (2.6). *Let μ, ν and η be positive Borel measures on $\mathbb{R}$, $\{\mu_\lambda\}$ be the family generated by μ. Then*

$$\eta(A) = \int_{\mathbb{R}} \mu_\lambda(A) d\nu(\lambda) \tag{2.7}$$

for every Borel set A if and only if

$$C\eta(z) = C\nu((-C\mu)^{-1}(z)). \tag{2.8}$$

Proof. $\Rightarrow$) Lemma 5.3 of [4].
$\Leftarrow$) By the definition of μ_λ, (2.8) implies

$$\int_{\mathbb{R}} \frac{d\eta(x)}{x-z} = \int_{\mathbb{R}} d\nu(\lambda) \int_{\mathbb{R}} \frac{d\mu_\lambda(x)}{x-z}$$

Since the kernels $(x-z)^{-1}$ and their conjugates span the space of continuous functions, we get (2.7) □

LEMMA (2.9). *Let μ and ν be positive Borel measures on $\mathbb{R}$, $\{\mu_\lambda\}$ and $\{\nu_\lambda\}$ be the families generated by these measures correspondingly. Define the measure η as*

$$\eta(A) = \int_{\mathbb{R}} \mu_\xi(A) d\nu(\xi) \tag{2.10}$$

for every Borel set A. Let η_λ be the family generated by η. Then

$$\eta_\lambda(A) = \int_{\mathbb{R}} \mu_\xi(A) d\nu_\lambda(\xi)$$

for all $\lambda \in \mathbb{R}$.

Proof. Definition (2.10) implies $C\eta = C\nu(-C_\mu^{-1})$ by the last lemma. Therefore

$$f_\lambda(C\eta(z)) = f_\lambda(C\nu(-C_\mu^{-1}(z)))$$

i.e.

$$C\eta_\lambda(z) = C\nu_\lambda(-C_\mu^{-1}(z))$$

and applying Lemma (2.6) again we obtain

$$\eta_\lambda(A) = \int \mu_\xi(A) d\nu_\lambda(\xi)$$

for every Borel set A, $\lambda \in \mathbb{R}$ □

Remark (2.11). If in the above proof one uses a simplified definition of the domain

$$\Gamma := \cap_n \Omega(E_n, \epsilon_n)$$

then the domain will not have a Dini–smooth corner of opening π at zero and the resulting measures μ_0 may not be finite. Hence the vector $\varphi = 1$ will not belong to the Hilbert space itself but to the space $\mathcal{H}_{-2}$ which is the dual of $\mathcal{H}_2$, where $\mathcal{H}_2$ is the same as the domain of A_0 with the graph norm

$$\| \psi \|_2 := \| (|A_0| + 1)\psi \|_{L_2}$$

In this case expression (1.1) has to be understood in a generalized way. The operator A_λ can be defined through

$$\frac{1}{A_\lambda - z} = \frac{1}{A - z} - \frac{1}{1/\lambda + c + \langle \varphi, \frac{1+Az}{A-z}\frac{1}{A^2+1}\varphi\rangle}\left\langle \frac{1}{A-\bar{z}}\varphi, \cdot \right\rangle \frac{1}{A-z}\varphi$$

where the parameter c may be chosen arbitrary in $\mathbb{R}$ for $\varphi \in \mathcal{H}_{-2}\backslash\mathcal{H}_{-1}$ and $c = \langle \varphi, \frac{A}{A^2+1}\varphi\rangle$ if $\varphi \in \mathcal{H}_{-1}$.

For definitions and more on the scales of Hilbert spaces associated to a self-adjoint operator we refer to [1, 17]. We thank a referee for very useful advice concerning these matters.

3. An example of coexistence of spectra

As was announced in the introduction, we now show that the words "almost every" can be ommitted in the statement of Theorem (1.2) when the set B is simple. The following observations will be useful in our construction.

Consider the family of Pick functions

$$f_\lambda(z) = \frac{z}{1+\lambda z} \quad \lambda \in \mathbb{R}$$

as in (2.5). Let $L(z)$ be a Pick function such that $0 \leq \text{Im } L(z) \leq \pi$.

For any $\lambda \in \mathbb{R}$ consider

$$\begin{aligned} M_\lambda(z) &= f_\lambda(L(z)) \\ N_\lambda(z) &= f_\lambda(H(L(z))) \end{aligned}$$

where

$$\begin{aligned} H(z) &= \exp(az) + c, \quad a, c \in \mathbb{R} \\ & \quad a \in (0, 1]. \end{aligned}$$

Both M_λ and N_λ are Pick functions admitting representations similiar to (2.1). We shall denote by μ_λ and ν_λ the measures appearing in the representations for M_λ and N_λ respectively.

As usual, we use the symbol γ^{ac} to denote the absolutely continuous part of a measure γ and γ^s for the singular part.

LEMMA (3.1). *Let μ_λ and ν_λ be defined as above and put*

$$\tilde{\lambda}(\lambda) \quad = -H^{-1}\left(-\tfrac{1}{\lambda}\right).$$

Then

$$\mu^s_\lambda = \tilde{\lambda}'(\lambda)\, \nu^s_{\tilde{\lambda}}.$$

Proof. We follow closely the proof of Lemma 2 in [3]. We shall first show that

$$\lim_{y \downarrow 0} \frac{\text{Im } M_\lambda(x+iy)}{\text{Im } N_{\tilde{\lambda}}(x+iy)} = \tilde{\lambda}'(\lambda)$$

for μ^s_λ-a.e. x. We use the notation $z = x + iy$.

First note that if λ and $\tilde{\lambda}$ are different from zero we have

$$\left(\frac{\tilde{\lambda}}{\lambda}\right)^2 \left|\frac{H(L(z)) - (-\frac{1}{\tilde{\lambda}})}{L(z) - (-\frac{1}{\lambda})}\right|^2 \cdot \frac{\text{Im } L(z)}{\text{Im } H(L(z))} = \frac{\text{Im } M_\lambda(z)}{\text{Im } N_{\tilde{\lambda}}(z)} \tag{3.2}$$

It is well known that for μ_λ^s-a.e. x.

$$\lim_{y \downarrow 0} M_\lambda(x + i\, y) = \infty$$

and therefore using (2.5) and recalling that $M_0(z) = L(z)$ we get

$$\lim_{y \downarrow 0} L(x + i\, y) = -\frac{1}{\lambda}$$

for μ_λ^s-a.e. x. Therefore using the definition of $\tilde{\lambda}$ we obtain for μ_λ^s-a.e. x.

$$\begin{aligned} \lim_{y \downarrow 0} \frac{H(L(x + i\, y)) - (-\frac{1}{\tilde{\lambda}})}{L(x + i\, y) - (-\frac{1}{\lambda})} &= \lim_{y \downarrow 0} \frac{H(L(x + i\, y)) - H(-\frac{1}{\lambda})}{L(x + i\, y) - (-\frac{1}{\lambda})} = \\ H'(-\frac{1}{\lambda}) &= a \exp a(-\frac{1}{\lambda}) \end{aligned} \tag{3.3}$$

Now

$$\begin{aligned} \lim_{y \downarrow 0} \frac{\text{Im } H(L(x + i\, y))}{\text{Im } L(x + i\, y)} &= \lim_{y \downarrow 0} \frac{\text{Im } \exp(a\, L(x + i\, y))}{\text{Im } L(x + i\, y)} = \\ \lim_{y \downarrow 0} \frac{\text{Im } \exp(a\, L(x + i\, y))}{\text{Im } L(x + i\, y)} &= a \exp a(-\frac{1}{\lambda}) \text{ for } \mu_\lambda^s - \text{a.e. } x \end{aligned} \tag{3.4}$$

Since $L(x + i\, 0) = -\frac{1}{\lambda}$ for such points x.

From (3.2, 3.3, 3.4) we get

$$\lim_{y \downarrow 0} \frac{\text{Im } M_\lambda(x + i\, y)}{\text{Im } N_{\tilde{\lambda}}(x + i\, y)} = \left(\frac{\tilde{\lambda}}{\lambda}\right)^2 (a \exp a(-\frac{1}{\lambda})) = \tilde{\lambda}'(\lambda)$$

Since the Poisson integral $P_{\mu_\lambda}(x + i\, y)$ tends to infinity at μ_λ^s-a.e. x the last equation implies

$$\lim_{y \downarrow 0} \frac{\text{Im } M_\lambda(x + i\, y)}{\text{Im } N_{\tilde{\lambda}}(x + i\, y)} = \frac{\text{Im } C\mu_\lambda(x + i\, y) + C_1 y}{\text{Im } C\nu_{\tilde{\lambda}}(x + i\, y) + C_2 y} = \lim_{y \downarrow 0} \frac{P_{\mu_\lambda}(z)}{P_{\nu_{\tilde{\lambda}}}(z)} = \tilde{\lambda}(\lambda)$$

From here the result easily follows as in Lemma 2 of [3]. □

Example (3.5). Now we are ready to show that there exists a self-adjoint operator A_0 such that the operators A_λ, defined as in (1.1), have dense singular spectrum for $\lambda \in \mathbb{R} \backslash [0, 1]$ and dense purely absolutely continuous spectrum for $\lambda \in [0, 1]$

Let $E \subset \mathbb{R}$ be such that for every interval $J \subset \mathbb{R}$ we have

$$|\, J \,| > |\, E \cap J \,| > 0$$

and such that $|E| < \infty$. To see examples of such sets we refer to [7].

Put

$$d\mu = \frac{1}{2} u(x)\, dx$$

where

$$u(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Let

$$C\mu(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}$$

and define

$$F(z) = \exp C\mu(z)$$

Since F is a Pick function it can be represented as in (2.1). To evaluate the constants a and b in (2.1) observe that $C\mu(z) \to 0$ when z goes to ∞ along the imaginary axis. Hence

$$F(z) = 1 + C\nu(z)$$

where $C\nu(z) = \int_{\mathbb{R}} \frac{d\,\nu(x)}{x-z}$ for some finite measure ν.

Once we have the measure ν we proceed to define the family of operators A_λ in the standard way: Let A_0 be the operator of multiplication by x in $L^2(\nu)$ and

$$A_\lambda = A + \lambda < \varphi, \cdot > \varphi$$

where $\varphi(x) = 1$ for every $x \in \mathbb{R}$.

We claim the family of operators A_λ has the desired properties. To see this observe that the measures corresponding to A_λ are the ones generated by the family of Pick functions (see [17]).

$$C\nu_\lambda(z) = f_\lambda(C\nu(z)) = f_\lambda(\exp C\mu(z) - 1) \tag{3.6}$$

From the definition of μ we have

$$C\mu(z) = \frac{1}{2} Cu(z)$$

where

$$Cu(z) = \int_{\mathbb{R}} \frac{u(x)dx}{\lambda - z}$$

We shall denote by u_λ the measure which appears in the integral representation of the Pick function $Cu_\lambda = f_\lambda(Cu(z))$.

According to Lemma (3.1) with $H(z) = \exp \frac{1}{2}z - 1$ we know that

$$u^s_\lambda(B) = \tilde{\lambda}'(\lambda)\, \nu^s_{\tilde{\lambda}}(B)$$

where

$$\tilde{\lambda}(\lambda) = (1 - \exp(-\frac{1}{2\lambda}))^{-1}$$

Therefore if

$$u^s_\lambda \text{ is densely supported for every } \lambda \neq 0 \tag{3.7}$$

then

$$\nu^s_{\tilde{\lambda}} \text{ is densely supported for } \tilde{\lambda} \in \mathbb{R} \backslash [0, 1]$$

To obtain (3.7) consider the Pick function $G = \exp Cu(z)$. Since

$$u(x) = \frac{1}{\pi} \arg(\exp Cu(x + i0))$$

we obtain that G has real boundary values a. e. on $\mathbb{R}$ and therefore the measure asociated with G is purely singular. From the stability of the absolutely continuous part it follows that the measures η_λ associated with the Pick functions $f_\lambda(\exp Cu(z))$ are purely singular too. By the choice of E the measure η_0 associated with G and therefore all the measures η_λ are densely supported.

Since by Lemma (3.1)

$$u_\lambda^s(J) = \tilde{\lambda}'(\lambda)\,\eta_{\tilde{\lambda}}^s(J)$$

where

$$\tilde{\lambda}(\lambda) = -(\exp(-\frac{1}{\lambda}))^{-1}$$

we get (3.7). Hence $\nu_{\tilde{\lambda}}^s$ is densely supported for every $\lambda \in \mathbb{R}\backslash[0,1]$ as required.

Now we shall see that if $\lambda \in [0,1]$ then the measure ν_λ is purely absolutely continuous.

Consider the following support of the singular part of the measure ν_λ

$$\Sigma_\lambda = \{x | C\nu_\lambda(x+i0) = \infty\}$$

see [11] for example, which can be seen to be equal to the set

$$\{x | C\nu_0(x+i0) = -\frac{1}{\lambda}\}$$

using the definition of the famiy f_λ.

From the definition (3.6) and the properties of the Poisson kernel it follows that

$$\begin{aligned}\arg\big(C\nu_0(x+i0)) + 1\big) = \arg\big(\exp C\mu(x+i0)\big) = \\ \operatorname{Im}\ C\mu(x+i0) = P\mu(x+i0) = \\ \frac{\pi}{2}\lim_{\epsilon\downarrow 0}\frac{|\,E\cap(x-\epsilon,x+\epsilon)\,|}{2\epsilon}.\end{aligned}$$

Since for $x \in \Sigma_\lambda$ $\;C\nu_0(x+i0) = -\frac{1}{\lambda} \in \mathbb{R}$ we should have

$$\lim_{\epsilon\downarrow 0}\frac{|\,E\cap(x-\epsilon,x+\epsilon)\,|}{2\epsilon} = 0 \text{ or } 2$$

and from the inequality

$$\frac{|\,E\cap(x-\epsilon,x+\epsilon)\,|}{2\epsilon} \le \frac{2\epsilon}{2\epsilon} = 1$$

it follows

$$\lim_{\epsilon\downarrow 0}\frac{|\,E\cap(x-\epsilon,x+\epsilon)\,|}{2\epsilon} = 0\,.$$

Therefore $-\frac{1}{\lambda} = \exp(\frac{1}{2}\operatorname{Re} C\mu) - 1$ which implies $\lambda \in \mathbb{R}\backslash[0,1]$. Thus if $\lambda \in [0,1]$ we have $\nu_{\tilde{\lambda}}^s = 0$.

Remark (3.8). In the case of a family of ergodic operators H_ω we can not have an analogous situation to the one described above taking instead of λ the parameter ω. This follows from a theorem of Kunz and Souillard see [2, p.169], which tells us that there exist sets $\sum_s$ and $\sum_{ac}$ such that

$$\begin{aligned} \sigma_s(H_\omega) &= \sum\nolimits_s \\ \sigma_{ac}(H_\omega) &= \sum\nolimits_{ac} \end{aligned}$$

for a.e. ω.

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R. del Rio
IIMAS-UNAM
Circuito Escolar, Ciudad Universitaria
04510, México, D.F.
México
delrio@servidor.unam.mx

S. Fuentes
Universidad Anahuac
Dpto Ciencias Basicas
Av. Univ. Anahuac s/n
Lomas Anahuac Huixquilucan
Edo. de Mexico
México
sergio.fuentes@anahuac.mx

A. Poltoratski
Department of Mathematics
Texas A & M University
College Station, Texas
77843–3368
U.S.A.
alexei.poltoratski@math.tamu.edu

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