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Instability of the Absolutely Continuous Spectrum of Ordinary Differential Operators under Local Perturbations

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It is shown that a potential q exists such that a selfadjoint realization of $lu = -u'' + q(x)u$ has singular continuous spectrum in an interval I while all the selfadjoint realizations of $\tilde{l}u = -u'' + \{q(x) + v(x)\}u$, where v is a continuous function with compact support, have absolutely continuous spectrum in I. © 1989 Academic Press, Inc.

1. INTRODUCTION

In J. Weidmann [9] it was proved that if L is a selfadjoint realization of the differential expression

$$(lu)(x) = -u''(x) + q(x)\,u(x), \qquad x \in (a, \infty),$$

where q is a real valued, locally integrable function defined in (a, ∞) and which satisfies certain conditions in the interval (c, ∞) with $c \in (a, \infty)$, then L has absolutely continuous spectrum in $(0, \infty)$ (see [2] for definition).

From this result it seems that the absolute continuity of the spectrum of L is determined by the behavior of the potential $q(x)$ for $x > c$. This suggests the following conjecture (which goes back essentially to J. Weidmann [9]): Let l be a formally selfadjoint differential expression in (a, b) and A a selfadjoint realization. For $c \in (a, b)$ let each selfadjoint realization A_b of l in (c, b) have absolutely continuous spectrum in $(\underline{\lambda}, \bar{\lambda})$. Then A also has absolutely continuous spectrum in $(\underline{\lambda}, \bar{\lambda})$. However, it was proved in [2] that this conjecture is false.

The above conjecture can be rewriten as follows supposing $a = 0$ is a regular point and $b = \infty$:

If the operator $\tilde{L}_\beta$ generated by

$$\tilde{l}u = -u''(x) + \tilde{q}(x)\,u(x), \qquad x \in [0, \infty),$$

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where

$$\tilde{q}(x) := q(x+c),$$

and the boundary condition

$$u(0)\cos\beta + u'(0)\sin\beta = 0,$$

has absolutely continuous spectrum in an interval I for all $\beta \in [0, 2\pi)$, then the operator L_α generated by

$$lu = -u(x)'' + q(x)\,u(x), \qquad x \in [0, \infty)$$

and

$$u(0)\cos\alpha + u'(0)\sin\alpha = 0$$

has also absolutely continuous spectrum in I.

Here $\tilde{q}$ can be considered as a perturbation of q and in general there will not exist any $p \in (0, \infty)$ such that $\tilde{q}(x) = q(x)$ for $x > p$. The perturbation $\tilde{q}$ affects q up to infinity, so to speak.

Now we make the same conjecture but for perturbations $\tilde{q}$ which are really local, that is to say, perturbations of the form

$$\tilde{q}(x) := q(x) + v(x),$$

where $v(x)$ is a function with compact support $S \subset (0, p)$.

The purpose of the present work is to show that also in this case the above conjecture is false, thus proving that only the behavior of the potential $q(x)$ near infinity cannot determine whether the spectrum is absolutely continuous or if it has a singular part.

2. Statement of the Main Result

We construct first an operator L with singular continuous spectrum in the following way. Let

$$\rho_i : \mathbf{R} \to \mathbf{R}, \qquad i = 1, 2$$

be non-decreasing functions such that

(a) $\rho_1(\lambda)$ is absolutely continuous in the interval $I \subset \mathbf{R}$,

$$\left.\frac{d\rho_1}{d\lambda}\right|_{\lambda \in I} \geqslant N > 0,$$

ρ_2 is singular continuous in I.

(b) The function $\rho := \rho_1 + \rho_2$ satisfies the hypotheses of the theorem of Gelfand and Levitan, (see [7]).

By (b) we know that there exists a differential operator L with spectral function ρ, defined through the differential expression

$$(lu)(x) = -u''(x) + q(x)\,u(x), \qquad 0 \leqslant x < \infty,$$

where $q: \mathbf{R}^+ \to \mathbf{R}$ is continuous, and the boundary condition

$$u(0)\cos\alpha + u'(0)\sin\alpha = 0.$$

The operator L is exactly the operator L_0 used in Section 3 of [2].

Here we note that the potential $q(x)$ cannot decay fast, as $x \to \infty$, otherwise the spectrum would be absolutely continuous; see, for example, [9].

Let $v: \mathbf{R}^+ \to \mathbf{R}$ be a continuous function with compact support $S \subset \mathbf{R}^+$.

Let us define the selfadjoint operator $\tilde{L}$ as the one generated by the differential expression

$$\tilde{l}u = -u'' + \{q(x) + v(x)\}\,u, \qquad x \in [0, \infty)$$

and the boundary condition

$$u(0)\cos\beta + u'(0)\sin\beta = 0, \qquad \beta \in [0, 2\pi).$$

Choose now $p \in \mathbf{R}$ such that $S \subset [0, p)$.

We define the operator L_α as the operator generated through the differential expression

$$lu = -u'' + q(x)u, \qquad x \in [0, p]$$

and the boundary conditions

$$u(0)\cos\alpha + u'(0)\sin\alpha = 0$$
$$u(p) = 0.$$

Similarly we define the operator L_β as the operator generated by the differential expression

$$\tilde{l}u = -u'' + \{q(x) + v(x)\}\,u, \qquad x \in [0, p]$$

and the boundary conditions

$$u(0)\cos\beta + u'(0)\sin\beta = 0$$
$$u(p) = 0, \qquad \beta \in [0, 2\pi).$$

L_α and L_β are selfadjoint operators generated by differential expressions which are regular in $[0, p]$ and therefore their spectra consist only of isolated eigenvalues.

Our main result is the following

THEOREM. *If L_α and L_β do not have exactly the same spectrum, then the operator $\tilde{L}$ has only absolutely continuous spectrum in I.*

If we remember that the operator L by construction has singular continuous spectrum in I, what the theorem says is that if the local perturbation $v(x)$ satisfies certain conditions, then the singular continuous spectrum disappears and we have pure absolutely continuous spectrum.

3. Some Lemmas

Before we prove the theorem we need some lemmas.

Consider a fundamental system $\{u_1(x, z), u_2(x, z)\}$ of solutions of

$$lu_k = -u_k''(x) + q(x)\, u_k(x) = zu_k(x), \qquad k = 1, 2, \qquad 0 \leqslant x < \infty$$

which satisfy the conditions

$$u_1(0, z) \cos \alpha + u_1'(0, z) \sin \alpha = 0$$
$$u_2(p, z) = 0$$
$$u_2'(p, z) = 1.$$

The point p is chosen as before, that is to say, p is to the right of the support of $v(x)$.

Consider also a fundamental system $\{\tilde{u}_1(x, z), \tilde{u}_2(x, z)\}$ of solutions of

$$\tilde{l}\tilde{u}_k = -\tilde{u}_k''(x) + \{q(x) + v(x)\}\, \tilde{u}_k = z\tilde{u}_k(x), \qquad k = 1, 2, \qquad 0 \leqslant x < \infty$$

such that $\tilde{u}_1$ and $\tilde{u}_2$ satisfy the conditions

$$\tilde{u}_1(0, z) \cos \beta + \tilde{u}_1'(0, z) \sin \beta = 0$$
$$\tilde{u}_2(p, z) = 0$$
$$\tilde{u}_2'(p, z) = 1.$$

It is known (see [1]) that if z is non-real, there is a function $m(z)$ such that

$$m(z)\, u_1(x, z) + u_2(x, z) \in L_2(0, \infty).$$

We call this function the Weyl–Titchmarsh–Kodaira coefficient (WTK henceforth) of L with respect to $\{u_1(x, z), u_2(x, z)\}$.

Let $\tilde{m}$ be the WTK coefficient of $\tilde{L}$ with respect to $\tilde{u}_1(x, z), \tilde{u}_2(x, z)$.

LEMMA 1. *Let $\lambda \in C$ be such that* Im $\lambda > 0$. *Then we have*

$$\tilde{m}(\lambda) = \frac{m(\lambda)}{C_1(\lambda) - C_2(\lambda)\, m(\lambda)}, \tag{1}$$

where $C_1(\lambda)$ and $C_2(\lambda)$ are analytic functions.

Proof. If Im $\lambda > 0$ it follows that $W(u_1, u_2)(\lambda) \neq 0$ and $W(\tilde{u}_1, \tilde{u}_2)(\lambda) \neq 0$ (where W denotes the Wronskian). Otherwise the selfadjoint operators L_α and L_β would have a non-real eigenvalue.

The WTK coefficients m and $\tilde{m}$ are given by

$$m(\lambda) = -\lim_{x \to \infty} \frac{u_2(x, \lambda)}{u_1(x, \lambda)}$$

and

$$\tilde{m}(\lambda) = -\lim_{x \to \infty} \frac{\tilde{u}_2(x, \lambda)}{\tilde{u}_1(x, \lambda)}.$$

Since u_2 and $\tilde{u}_2$ are solutions of the same equation for $x > p$ and satisfy the same conditions at p it follows that

$$u_2(x, \lambda) \equiv \tilde{u}_2(x, \lambda) \qquad \text{when} \quad x > p.$$

Therefore we have

$$\tilde{m}(\lambda) = -\lim_{x \to \infty} \frac{u_2(x, \lambda)}{\tilde{u}_1(x, \lambda)}.$$

Since $\tilde{u}_1(x, \lambda)$ is a solution of $lu = \lambda u$ when $x > p$ it follows that

$$\tilde{u}_1(x, \lambda) = C_1(\lambda)\, u_1(x, \lambda) + C_2(\lambda)\, u_2(x, \lambda) \qquad \text{if } x > p.$$

Therefore

$$\tilde{m}(\lambda) = -\lim_{x \to \infty} \frac{u_2(x, \lambda)}{C_1(\lambda)\, u_1(x, \lambda) + C_2(\lambda)\, u_2(x, \lambda)}.$$

Dividing by u_1 and taking the limit follows (1).

The analyticity of $C_1(\lambda)$ and $C_2(\lambda)$ is a consequence of the analyticity of $u_1(x, \lambda)$, $u_2(x, \lambda)$, and $\tilde{u}_1(x, \lambda)$ with respect to λ. Q.E.D.

LEMMA 2. *If L_α and L_β do not have exactly the same spectrum we have*

$$W(\tilde{u}_1, u_1)(x, \lambda) \not\equiv 0$$

(W denotes the Wronskian).

Proof. Suppose that $W(\tilde{u}_1, u_1)(x, \lambda) \equiv 0$ for $\lambda \in \mathbf{R}$ and $x \geqslant p$. It follows that for $x \geqslant p$, $\lambda \in \mathbf{R}$,

$$\tilde{u}_1(x, \lambda) = k(\lambda)\, u_1(x, \lambda)$$
$$k(\lambda) \neq 0.$$

We know from the hypotheses that

$$u_2(p, \lambda) = 0 = \tilde{u}_2(p, \lambda)$$
$$u_2'(p, \lambda) = 1 = \tilde{u}_2'(p, \lambda).$$

Therefore,

$$W(u_1, u_2)(p, \lambda) = u_1(p, \lambda)$$

and

$$W(\tilde{u}_1, \tilde{u}_2)(p, \lambda) = \tilde{u}_1(p, \lambda).$$

Hence

$$W(\tilde{u}_1, \tilde{u}_2)(p, \lambda) = k(\lambda)\, W(u_1, u_2)(p, \lambda).$$

Now, the Wronskian is the same for all $x \in (0, p)$, so we can write

$$W(\tilde{u}_1, \tilde{u}_2)(\lambda) = k(\lambda)\, W(u_1, u_2)(\lambda)$$

for $\lambda \in \mathbf{R}$. This implies that the selfadjoint operators L_α and L_β have the same eigenvalues and we have reached a contradiction. Q.E.D.

Let A be the set of points $\lambda_i \in I$ such that $W(u_1, u_2)(\lambda_i) = 0$, points $\mu_i \in I$ such that $W(\tilde{u}_1, u_1)(\mu_i) = 0$, and points $\gamma_i \in I$ such that $W(\tilde{u}_1, \tilde{u}_2)(\gamma_i) = 0$.

LEMMA 3. *For $u \in I' = I \setminus A$ there exist $N(u) > 0$ and $r(u) > 0$ such that*

$$|C_1(u + i\varepsilon) - C_2(u + i\varepsilon)\, m(u + i\varepsilon)| \geqslant N(u) > 0$$

holds, whenever $0 < \varepsilon < r(u)$.

Proof. By the definition of C_1 and C_2 (see Lemma 1 above) we have

$$\tilde{u}_1(x, \lambda) = C_1(\lambda)\, u_1(x, \lambda) + C_2(\lambda)\, u_2(x, \lambda)$$

(when $x > p$). It then follows that

$$C_1(\lambda) = \frac{W(\tilde{u}_1, u_2)}{W(u_1, u_2)}(\lambda)$$

$$C_2(\lambda) = \frac{W(\tilde{u}_1, u_1)}{W(u_2, u_1)}(\lambda)$$

from which we conclude that C_1 and C_2 are analytic if $\tilde{u}_1, u_1, u_2$ are.

Suppose now that $\lambda = u + i\varepsilon$, $\varepsilon > 0$, is such that

$$W(\tilde{u}_1, u_1)(\lambda) \neq 0.$$

Then,

$$\left| \frac{C_1(\lambda)}{W(\tilde{u}_1, u_1)(\lambda)} + \frac{m(\lambda)}{W(u_1, u_2)(\lambda)} \right| \geqslant \left| \operatorname{Im} \frac{m(\lambda)}{W(u_1, u_2)(\lambda)} \right| - \left| \operatorname{Im} \frac{C_1(\lambda)}{W(\tilde{u}_1, u_1)(\lambda)} \right|.$$

The function ρ_1 in addition to being absolutely continuous has by hypothesis (a) the property

$$\left. \frac{d\rho_1}{d\lambda} \right|_{\lambda \in I} \geqslant N > 0.$$

This implies that for $u \in I'$, $0 < \varepsilon < k'$, k' small enough,

$$\int_{-\infty}^{\infty} \frac{\varepsilon}{(\mu - u)^2 + \varepsilon^2} \, d\rho_1(\mu) \geqslant K > 0$$

and we conclude that there is a constant $k'' > 0$ such that if $0 < \varepsilon < k''$ then

$$-\operatorname{Im} \frac{m}{W(u_1, u_2)}(u + i\varepsilon) \geqslant K > 0$$

for $u \in I'$. (See [2].)

Let us note that $C_1(\lambda)$ and $W(\tilde{u}_1, u_1)(\lambda)$ are real whenever λ is real. Therefore

$$\left| \operatorname{Im} \frac{m(u + i\varepsilon)}{W(u_1, u_2)(u + i\varepsilon)} \right| - \left| \operatorname{Im} \frac{C_1(u + i\varepsilon)}{W(\tilde{u}_1, u_1)(u + i\varepsilon)} \right| \geqslant N(u) > 0$$

if $\varepsilon < k(u)$, $u \in I'$, k small enough.

Now, we are assuming that $\lambda = u + i\varepsilon$, $\varepsilon > 0$, is such that $W(\tilde{u}_1, u_1)(\lambda) \neq 0$, therefore

$$|W(\tilde{u}_1, u_1)(\lambda)| \left| \frac{C_1(\lambda)}{W(\tilde{u}_1, u_1)(\lambda)} + \frac{m(\lambda)}{W(u_1, u_2)(\lambda)} \right| \geqslant |W(\tilde{u}_1, u_1)(\lambda)| \, N(u) > 0$$

if $0 < \varepsilon < k(u)$.

If we recall the form of C_2 we obtain

$$|C_1(\lambda) - C_2(\lambda)\, m(\lambda)| \geqslant |W(\tilde{u}_1, u_1)(\lambda)|\, N(u) \qquad \text{if} \quad 0 < \varepsilon < k(u).$$

Since $W(\tilde{u}_1, u_1)(u) \neq 0$ because $u \in I'$ we can take a closed ball $B_u(r)$ with center in u and radius r small enough, in particular $r < k(u)$, such that $W(\tilde{u}_1, u_1)(\lambda) \neq 0$ for every $\lambda \in B_u(r)$.

Since $W(\tilde{u}_1, u_1)(\lambda)$ is analytic it reaches its minimum M in $B_u(r)$. Hence

$$|C_1(u+i\varepsilon) - C_2(u+i\varepsilon)\, m(u+i\varepsilon)| \geqslant MN(u) = N(u) > 0$$

if $0 < \varepsilon < r(u)$ for $u \in I'$. Q.E.D.

With the help of the preceding lemmas we shall prove the following result. Remember that $\tilde{m}$ denotes the WTK coefficient of $\tilde{L}$ with respect to $\tilde{u}_1(x, z)$, $\tilde{u}_2(x, z)$.

LEMMA 4. *If the operators L_α and L_β do not have exactly the same spectrum then it is not possible that*

$$\lim_{\varepsilon \downarrow 0} |\tilde{m}(u+i\varepsilon)| = \infty$$

for $u \in I' = I \setminus A$.

Proof. Suppose we have

$$\lim_{\varepsilon \downarrow 0} |\tilde{m}(u+i\varepsilon)| = \infty.$$

Using Lemma 1 we have

$$\lim_{\varepsilon \downarrow 0} \left| \frac{m(\lambda)}{C_1(\lambda) - C_2(\lambda)\, m(\lambda)} \right| = \infty,$$

where $\lambda = u + i\varepsilon$, $u \in I'$.

From Lemma 3 we know that we can choose $k(u)$ small enough so that if $0 < \varepsilon < k(u)$, $u \in I'$, then

$$|C_1(u+i\varepsilon) - C_2(u+i\varepsilon)\, m(u+i\varepsilon)| \geqslant N(u) > 0.$$

Moreover, since we are supposing that $\lim_{\varepsilon \downarrow 0} |\tilde{m}(u+i\varepsilon)| = \infty$, we can choose $k(u)$ small enough so that

$$|\tilde{m}(u+i\varepsilon)| > M > 0 \qquad \text{if} \quad 0 < \varepsilon < k(u)$$

holds, where $M > 0$ is a given arbitrary constant.

Therefore if $0<\varepsilon<k(u)$ then

$$|m(u+i\varepsilon)| \geqslant N(u)\, M>0.$$

Since M is arbitrary it follows that

$$\lim_{\varepsilon \downarrow 0} |m(u+i\varepsilon)| = \infty \qquad \text{for} \quad u \in I'.$$

Now, if $|m(\lambda)| \neq 0$ we have

$$|\tilde{m}(\lambda)| = \frac{1}{\left|\dfrac{C_1(\lambda)}{|m(\lambda)|} - \dfrac{C_2(\lambda)\, m(\lambda)}{|m(\lambda)|}\right|}$$

$$\leqslant \frac{1}{|C_2(\lambda)| - \left|\dfrac{C_1(\lambda)}{m(\lambda)}\right|}$$

therefore

$$\lim_{\varepsilon \downarrow 0} |\tilde{m}(u+i\varepsilon)| \leqslant \lim_{\varepsilon \downarrow 0} \frac{1}{|C_2(u+i\varepsilon)| - \left|\dfrac{C_1(u+i\varepsilon)}{m(u+i\varepsilon)}\right|} = \frac{1}{|C_2(u)|}$$

$$= \left|\frac{W(u_2, u_1)}{W(\tilde{u}_1, u_1)}(u)\right| < \infty$$

since if $u \in I'$ we have $W(\tilde{u}_1, u_1) \neq 0$. But this contradicts the assumption

$$\lim_{\varepsilon \downarrow 0} |\tilde{m}(u+i\varepsilon)| = \infty. \qquad \text{Q.E.D.}$$

4. Proof of the Theorem

Let $\tilde{\rho}$ be the spectral function of the operator $\tilde{L}$. We shall prove first that $\tilde{\rho}$ is continuous in I.

Let ${}_\theta L$ be the selfadjoint operator generated by the differential expression

$$(lu)(x) = -u''(x) + q(x)\, u(x), \qquad 0 \leqslant x < \infty$$

and the boundary condition

$$u(0) \cos\theta + u'(0) \sin\theta = 0, \qquad \theta \in [0, 2\pi).$$

(In particular ${}_\alpha L = L$.)

LEMMA 5. *The operators* ${}_\theta L$ *have only absolutely continuous spectrum in* I *if* $\theta \neq \alpha$.

For the proof of this lemma see [3].

LEMMA 6. *The spectral function* $\tilde{\rho}$ *of the operator* $\tilde{L}$ *is continuous in* I.

Proof. We shall prove that

$$\tilde{l}u = \lambda u, \qquad \lambda \in I \tag{2}$$

does not have L_2 solutions.

To prove this suppose that v is a solution of (2) and that v belongs to $L_2(0, \infty)$. Then we have that

$$(\tilde{l}v)(x) = (lv)(x) = \lambda v(x)$$

if $x \geqslant p$ and $v \in L_2(p, \infty)$.

Let $\{v_1, v_2\}$ be a system of solution of $lv_k = zv_k$ for $x \in [0, \infty)$ which satisfy

$$v_1(p, z) = 1, \qquad v_2(p, z) = 0$$

$$v_1'(p, z) = 0, \qquad v_2'(p, z) = 1$$

for all $z \in \mathbf{C}$.

There exist $C_1(\lambda)$, $C_2(\lambda) \in \mathbf{C}$ such that

$$h(x, \lambda) := C_1(\lambda)\, v_1(x, \lambda) + C_2(\lambda)\, v_2(x, \lambda) = v(x, \lambda) \qquad \text{for} \quad x \in [p, \infty).$$

It follows that $h(x) \in L_2(0, \infty)$ and that

$$(lh)(x) = \lambda h(x).$$

But Lemma 5 implies that $lu = \lambda u$ does not have solutions in $L_2(0, \infty)$ when $\lambda \in I$. Therefore we have a contradiction and (2) does not have L_2 solutions. This implies that the spectral function $\tilde{\rho}$ is continuous (see [4]). Q.E.D.

The fact that (2) does not have L_2 solutions implies (see [4]) that I is contained in the continuous spectrum of any selfadjoint realization of $\tilde{l}$ in $L_2(0, \infty)$.

Now let us define the symmetric derivative of the spectral function of the operator $\tilde{L}$

$$D\tilde{\rho}(u) := \lim_{\eta \to 0} \frac{\tilde{\rho}(u + \eta) - \tilde{\rho}(u - \eta)}{2\eta}.$$

LEMMA 7. *Suppose that L_α and L_β do not have exactly the same spectrum. If $D\tilde{\rho}(u)$ exists, then*

$$|D\tilde{\rho}(u)| < \infty \qquad \textit{for} \quad u \in I' = I \backslash A.$$

Proof. We know that

$$-\operatorname{Im} \frac{\tilde{m}(u+i\varepsilon)}{W(\tilde{u}_1, \tilde{u}_2)(u+i\varepsilon)} - \operatorname{Im} H(u+i\varepsilon) = \int_{-\infty}^{\infty} \frac{\varepsilon}{(u-\mu)^2+\varepsilon^2}\, d\tilde{\rho}(\mu),$$

where H is an analytic function. $H(z)$ is real if z real. See [2].

Furthermore we have that

$$\int_{-\infty}^{\infty} \frac{\varepsilon}{(u-\mu)^2+\varepsilon^2}\, d\tilde{\rho}(\mu) \geqslant \frac{1}{\varepsilon} \int_{u-\varepsilon}^{u+\varepsilon} \frac{\varepsilon^2}{(u-\mu)^2+\varepsilon^2}\, d\tilde{\rho}(\mu) \geqslant \frac{\tilde{\rho}(u+\varepsilon)-\tilde{\rho}(u-\varepsilon)}{2\varepsilon}.$$

Therefore

$$\left| \frac{\tilde{m}(u+i\varepsilon)}{W(\tilde{u}_1, \tilde{u}_2)(u+i\varepsilon)} \right| - \operatorname{Im} H(u+i\varepsilon) \geqslant \frac{\tilde{\rho}(u+\varepsilon)-\tilde{\rho}(u-\varepsilon)}{2\varepsilon} \geqslant 0.$$

From this inequality we see that if $D\tilde{\rho}(u) = \infty$ then

$$\lim_{\varepsilon \downarrow 0} |\tilde{m}(u+i\varepsilon)| = \infty \qquad \text{for} \quad u \in I',$$

which contradicts Lemma 4. Q.E.D.

If $\tilde{\rho}'(\lambda) := \lim_{h \to 0} ((\tilde{\rho}(\lambda+h) - \tilde{\rho}(\lambda))/h)$ exists, then $D\tilde{\rho}(\lambda)$ exists and $D\tilde{\rho}(\lambda) = \tilde{\rho}'(\lambda)$. Therefore by Lemma 7 above $\tilde{\rho}'(\lambda) < \infty$ if $\lambda \in I'$.

Proof of the Theorem. Let $E_\infty \subset I$ be the set of points λ where $\tilde{\rho}'(\lambda) = \infty$. Let us denote also by $\tilde{\rho}$ the measure associated with the monotone non-decreasing function $\tilde{\rho}$. By a theorem of de la Valleé-Poussin (see [8]) we know that if $X \subset I$ is a measurable set then

$$\tilde{\rho}(X) = \tilde{\rho}(X \cap E_\infty) + \int_X \tilde{\rho}'(x)\, dx.$$

Since $\tilde{\rho}$ is continuous by Lemma 6 and $E_\infty \subset A$, where A is a finite set by Lemma 7, it follows that $\tilde{\rho}$ is absolutely continuous in I and the theorem is proved. Q.E.D.

The theorem requires L_α and L_β not to have exactly the same spectrum. Now we shall construct $v: \mathbf{R}^+ \to \mathbf{R}$ continuous and with compact support $S \subset \mathbf{R}$ such that for every $\beta \in [0, 2\pi)$ this condition is satisfied.

We know that there exists an increasing unbounded sequence

$\lambda_0, \lambda_1, ..., \lambda_n, ...$ of eigenvalues of L_α and that the eigenfunction corresponding to the n-th eigenvalue has exactly n zeros in the open interval $(0, p)$. (See, for example, [6].)

Choose λ_n such that $n \geqslant 4$ and

$$\lambda_n > M = \sup_{x \in (0, p)} |q(x)|.$$

Choose an open interval $J = (a, b) \subset (0, p)$ such that the n zeros of the eigenfunction corresponding to λ_n are contained in J and let $k > \lambda_n$.

Let $v: \mathbf{R}^+ \to \mathbf{R}$ be a continuous function satisfying

(a) $v(x) = k - q(x)$ if $x \in J$,

(b) $v(x) > 0$ if $x \in (0, p)$,

(c) $v(x) = 0$ if $x \in \mathbf{R}^+ \setminus (0, p)$

(in particular $v(0) = v(p) = 0$).

Define then

$$\tilde{q}(x) := q(x) + v(x).$$

Using the comparison theorem (see [1, 6]) we can prove that the solutions of

$$-u'' + (\tilde{q}(x) - \lambda_n)\, u = 0$$

have at most three zeros in $(0, p)$, proving that the n-th eigenvalue of L_α cannot be the n-th eigenvalue of L_β.

Therefore we have constructed a perturbation which satisfies the hypothesis of the theorem for every $\beta \in [0, 2\pi)$.

Now let L be the operator

$$Lu \equiv -u'' + qu \tag{3}$$

subject to the boundary conditions

$$u(0) \cos \alpha + u'(0) \sin \alpha = 0 \tag{4}$$

$$u(1) \cos \beta + u'(1) \sin \beta = 0, \qquad \alpha, \beta \in [0, \pi). \tag{5}$$

In Hochstadt and Lieberman [5] the following result is proved:

THEOREM (See [5]). *Consider the operator* (3) *subject to* (4) *and* (5) *where q is summable on* $(0, 1)$. *Let* $\{\lambda_n\}$ *be the spectrum of L subject to* (4) *and* (5).

Consider a second operator

$$\tilde{L}u \equiv -u'' + \tilde{q}u,$$

where $\tilde{q}$ is summable on (0, 1) *and*

$$\tilde{q}(x) = q(x) \qquad on \quad (\tfrac{1}{2}, 1).$$

Suppose that the spectrum of $\tilde{L}$ subject to (4) *and* (5) *is also* $\{\lambda_n\}$. *Then* $q(x) = \tilde{q}(x)$ *almost everywhere on* (0, 1).

By scaling we can take $(0, p)$ instead of (0, 1) and $\tilde{q}(x) = q(x)$ on $(\frac{1}{2}p, p)$ instead of $\tilde{q}(x) = q(x)$ on $(\frac{1}{2}, 1)$.

With the help of this theorem if we choose $p \in \mathbf{R}$ such that $S \subset (0, \frac{1}{2}p)$, where S is the support of the perturbation $v(x)$, then it follows that L_α and L_β do not have the same spectrum when $\alpha = \beta$, unless $v(x) \equiv 0$. Then by the theorem proved in this work it follows that *every* perturbation $v(x)$ continuous and not identically null changes singular continuous spectrum into absolutely continuous spectrum.

We have proved therefore the following result, where $I \subset \mathbf{R}$ is an interval.

COROLLARY. *There exists a continuous potential $q(x)$ and $\alpha \in [0, \pi)$ such that*

$$lu = -u'' + q(x)u, \qquad x \in [0, \infty)$$

$$u(0) \cos \alpha + u'(0) \sin \alpha = 0$$

has singular continuous spectrum in I and such that for every continuous function $v(x)$, not identically null and with compact support, the operator generated by

$$\tilde{l}u = -u'' + \{q(x) + v(x)\}u, \qquad x \in [0, \infty)$$

and the boundary condition

$$u(0) \cos \alpha + u'(0) \sin \alpha = 0$$

has only absolutely continuous spectrum in I.

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