Instability of the Absolutely Continuous Spectrum of Ordinary Differential Operators under Local Perturbations

RAFAEL RENÉ DEL RÍO CASTILLO

Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas,
Universidad Nacional Autónoma de México,
Apto. Postal 20-726, Admón. No. 20,
Deleg. Alvaro Obregón, 01000 México, D. F., México

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It is shown that a potential $q$ exists such that a selfadjoint realization of $Lu = -u'' + q(x)u$ has singular continuous spectrum in an interval $I$ while all the selfadjoint realizations of $\tilde{L}u = -u'' + \{ q(x) + v(x) \} u$, where $v$ is a continuous function with compact support, have absolutely continuous spectrum in $I$. © 1989 Academic Press, Inc.

1. INTRODUCTION

In J. Weidmann [9] it was proved that if $L$ is a selfadjoint realization of the differential expression

$$(lu)(x) = -u''(x) + q(x)u(x), \quad x \in (a, \infty),$$

where $q$ is a real valued, locally integrable function defined in $(a, \infty)$ and which satisfies certain conditions in the interval $(c, \infty)$ with $c \in (a, \infty)$, then $L$ has absolutely continuous spectrum in $(0, \infty)$ (see [2] for definition).

From this result it seems that the absolute continuity of the spectrum of $L$ is determined by the behavior of the potential $q(x)$ for $x > c$. This suggests the following conjecture (which goes back essentially to J. Weidmann [9]): Let $l$ be a formally selfadjoint differential expression in $(a, b)$ and $A$ a selfadjoint realization. For $c \in (a, b)$ let each selfadjoint realization $A_{\beta}$ of $l$ in $(c, b)$ have absolutely continuous spectrum in $(\tilde{\lambda}, \tilde{\lambda})$. Then $A$ also has absolutely continuous spectrum in $(\tilde{\lambda}, \tilde{\lambda})$. However, it was proved in [2] that this conjecture is false.

The above conjecture can be rewritten as follows supposing $a = 0$ is a regular point and $b = \infty$:

If the operator $L_{\beta}$ generated by

$$\tilde{L}u = -u''(x) + \tilde{q}(x)u(x), \quad x \in [0, \infty),$$

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where
\[ \tilde{q}(x) := q(x + c), \]
and the boundary condition
\[ u(0) \cos \beta + u'(0) \sin \beta = 0, \]
has absolutely continuous spectrum in an interval \( I \) for all \( \beta \in [0, 2\pi) \), then the operator \( L_x \) generated by
\[ hu = -u(x)'' + q(x) u(x), \quad x \in [0, \infty) \]
and
\[ u(0) \cos \alpha + u'(0) \sin \alpha = 0 \]
has also absolutely continuous spectrum in \( I \).

Here \( \tilde{q} \) can be considered as a perturbation of \( q \) and in general there will not exist any \( p \in (0, \infty) \) such that \( \tilde{q}(x) = q(x) \) for \( x > p \). The perturbation \( \tilde{q} \) affects \( q \) up to infinity, so to speak.

Now we make the same conjecture but for perturbations \( \tilde{q} \) which are really local, that is to say, perturbations of the form
\[ \tilde{q}(x) := q(x) + v(x), \]
where \( v(x) \) is a function with compact support \( S \subset (0, p) \).

The purpose of the present work is to show that also in this case the above conjecture is false, thus proving that only the behavior of the potential \( q(x) \) near infinity cannot determine whether the spectrum is absolutely continuous or if it has a singular part.

2. **Statement of the Main Result**

We construct first an operator \( L \) with singular continuous spectrum in the following way. Let
\[ \rho_i : \mathbb{R} \to \mathbb{R}, \quad i = 1, 2 \]
be non-decreasing functions such that
(a) \( \rho_1(\lambda) \) is absolutely continuous in the interval \( I \subset \mathbb{R} \),
\[ \left. \frac{d\rho_1}{d\lambda} \right|_{\lambda \in I} \geq N > 0, \]
\( \rho_2 \) is singular continuous in \( I \).
(b) The function $\rho := \rho_1 + \rho_2$ satisfies the hypotheses of the theorem of Gelfand and Levitan, (see [7]).

By (b) we know that there exists a differential operator $L$ with spectral function $\rho$, defined through the differential expression

$$(\hat{u})(x) = -u''(x) + q(x)u(x), \quad 0 \leq x < \infty,$$

where $q: \mathbb{R}^+ \to \mathbb{R}$ is continuous, and the boundary condition

$$u(0) \cos x + u'(0) \sin x = 0.$$

The operator $L$ is exactly the operator $L_0$ used in Section 3 of [2].

Here we note that the potential $q(x)$ cannot decay fast, as $x \to \infty$, otherwise the spectrum would be absolutely continuous; see, for example, [9].

Let $v: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with compact support $S \subset \mathbb{R}^+$. Let us define the selfadjoint operator $\tilde{L}$ as the one generated by the differential expression

$$\tilde{u} = -u'' + \{q(x) + v(x)\}u, \quad x \in [0, \infty)$$

and the boundary condition

$$u(0) \cos \beta + u'(0) \sin \beta = 0, \quad \beta \in [0, 2\pi).$$

Choose now $p \in \mathbb{R}$ such that $S \subset [0, p)$.

We define the operator $L_x$ as the operator generated through the differential expression

$$lu = -u'' + q(x)u, \quad x \in [0, p]$$

and the boundary conditions

$$u(0) \cos x + u'(0) \sin x = 0$$
$$u(p) = 0.$$

Similarly we define the operator $L_\beta$ as the operator generated by the differential expression

$$\tilde{u} = -u'' + \{q(x) + v(x)\}u, \quad x \in [0, p]$$

and the boundary conditions

$$u(0) \cos \beta + u'(0) \sin \beta = 0$$
$$u(p) = 0, \quad \beta \in [0, 2\pi).$$
$L_\alpha$ and $L_\beta$ are selfadjoint operators generated by differential expressions which are regular in $[0, p]$ and therefore their spectra consist only of isolated eigenvalues.

Our main result is the following

**Theorem.** If $L_\alpha$ and $L_\beta$ do not have exactly the same spectrum, then the operator $\hat{L}$ has only absolutely continuous spectrum in $I$.

If we remember that the operator $L$ by construction has singular continuous spectrum in $I$, what the theorem says is that if the local perturbation $v(x)$ satisfies certain conditions, then the singular continuous spectrum disappears and we have pure absolutely continuous spectrum.

3. **Some Lemmas**

Before we prove the theorem we need some lemmas. Consider a fundamental system $\{u_1(x, z), u_2(x, z)\}$ of solutions of

$$lu_k = -u_k''(x) + g(x) u_k(x) = z u_k(x), \quad k = 1, 2, \quad 0 \leq x < \infty$$

which satisfy the conditions

$$u_1(0, z) \cos \alpha + u_1'(0, z) \sin \alpha = 0$$

$$u_2(p, z) = 0$$

$$u_2'(p, z) = 1.$$

The point $p$ is chosen as before, that is to say, $p$ is to the right of the support of $v(x)$.

Consider also a fundamental system $\{\tilde{u}_1(x, z), \tilde{u}_2(x, z)\}$ of solutions of

$$\tilde{l}\tilde{u}_k = -\tilde{u}_k''(x) + \{g(x) + v(x)\} \tilde{u}_k = z\tilde{u}_k(x), \quad k = 1, 2, \quad 0 \leq x < \infty$$

such that $\tilde{u}_1$ and $\tilde{u}_2$ satisfy the conditions

$$\tilde{u}_1(0, z) \cos \beta + \tilde{u}_1'(0, z) \sin \beta = 0$$

$$\tilde{u}_2(p, z) = 0$$

$$\tilde{u}_2'(p, z) = 1.$$

It is known (see [1]) that if $z$ is non-real, there is a function $m(z)$ such that

$$m(z) u_1(x, z) + u_2(x, z) \in L_2(0, \infty).$$
We call this function the Weyl–Titchmarsh–Kodaira coefficient (WTK henceforth) of $L$ with respect to \{u_1(x, z), u_2(x, z)\}.

Let $\tilde{m}$ be the WTK coefficient of $\tilde{L}$ with respect to $\tilde{u}_1(x, z), \tilde{u}_2(x, z)$.

**Lemma 1.** Let $\lambda \in \mathbb{C}$ be such that $\text{Im} \, \lambda > 0$. Then we have

$$\tilde{m} (\lambda) = \frac{m (\lambda)}{C_1 (\lambda) - C_2 (\lambda) m (\lambda)},$$  \hspace{1cm} (1)

where $C_1 (\lambda)$ and $C_2 (\lambda)$ are analytic functions.

**Proof.** If $\text{Im} \, \lambda > 0$ it follows that $W(u_1, u_2)(\lambda) \neq 0$ and $W(\tilde{u}_1, \tilde{u}_2)(\lambda) \neq 0$ (where $W$ denotes the Wronskian). Otherwise the selfadjoint operators $L_x$ and $L_\beta$ would have a non-real eigenvalue.

The WTK coefficients $m$ and $\tilde{m}$ are given by

$$m (\lambda) = \lim_{x \to \infty} \frac{u_2(x, \lambda)}{u_1(x, \lambda)}$$

and

$$\tilde{m} (\lambda) = \lim_{x \to \infty} \frac{\tilde{u}_2(x, \lambda)}{\tilde{u}_1(x, \lambda)}.$$

Since $u_2$ and $\tilde{u}_2$ are solutions of the same equation for $x > p$ and satisfy the same conditions at $p$ it follows that

$$u_2(x, \lambda) \equiv \tilde{u}_2(x, \lambda) \quad \text{when} \quad x > p.$$

Therefore we have

$$\tilde{m} (\lambda) = \lim_{x \to \infty} \frac{u_2(x, \lambda)}{\tilde{u}_1(x, \lambda)}.$$

Since $\tilde{u}_1(x, \lambda)$ is a solution of $lu = \lambda u$ when $x > p$ it follows that

$$\tilde{u}_1(x, \lambda) = C_1 (\lambda) u_1(x, \lambda) + C_2 (\lambda) u_2(x, \lambda) \quad \text{if} \quad x > p.$$

Therefore

$$\tilde{m} (\lambda) = \lim_{x \to \infty} \frac{u_2(x, \lambda)}{C_1 (\lambda) u_1(x, \lambda) + C_2 (\lambda) u_2(x, \lambda)}.$$

Dividing by $u_1$ and taking the limit follows (1).

The analyticity of $C_1 (\lambda)$ and $C_2 (\lambda)$ is a consequence of the analyticity of $u_1(x, \lambda), u_2(x, \lambda)$, and $\tilde{u}_1(x, \lambda)$ with respect to $\lambda$.

Q.E.D.
Lemma 2. If \( L_\alpha \) and \( L_\beta \) do not have exactly the same spectrum we have
\[
W(\tilde{u}_1, u_1)(x, \lambda) \neq 0
\]
(\( W \) denotes the Wronskian).

Proof. Suppose that \( W(\tilde{u}_1, u_1)(x, \lambda) \equiv 0 \) for \( \lambda \in \mathbb{R} \) and \( x \geq p \). It follows that for \( x \geq p, \lambda \in \mathbb{R}, \)
\[
\tilde{u}_1(x, \lambda) = k(\lambda) u_1(x, \lambda)
\]
\[
k(\lambda) \neq 0.
\]
We know from the hypotheses that
\[
u_2(p, \lambda) = 0 = \tilde{u}_2(p, \lambda)
\]
\[
u'_2(p, \lambda) = 1 = \tilde{u}'_2(p, \lambda).
\]
Therefore,
\[
W(u_1, u_2)(p, \lambda) = u_1(p, \lambda)
\]
and
\[
W(\tilde{u}_1, \tilde{u}_2)(p, \lambda) = \tilde{u}_1(p, \lambda).
\]
Hence
\[
W(\tilde{u}_1, \tilde{u}_2)(p, \lambda) = k(\lambda) W(u_1, u_2)(p, \lambda).
\]
Now, the Wronskian is the same for all \( x \in (0, p) \), so we can write
\[
W(\tilde{u}_1, \tilde{u}_2)(\lambda) = k(\lambda) W(u_1, u_2)(\lambda)
\]
for \( \lambda \in \mathbb{R} \). This implies that the selfadjoint operators \( L_\alpha \) and \( L_\beta \) have the same eigenvalues and we have reached a contradiction. Q.E.D.

Let \( A \) be the set of points \( \lambda_i \in I \) such that \( W(u_1, u_2)(\lambda_i) = 0 \), points \( \mu_i \in I \) such that \( W(\tilde{u}_1, u_1)(\mu_i) = 0 \), and points \( \gamma_i \in I \) such that \( W(\tilde{u}_1, \tilde{u}_2)(\gamma_i) = 0 \).

Lemma 3. For \( u \in I' = I \backslash A \) there exist \( N(u) > 0 \) and \( r(u) > 0 \) such that
\[
|C_1(u + i\varepsilon) - C_2(u + i\varepsilon) m(u + i\varepsilon)| \geq N(u) > 0
\]
holds, whenever \( 0 < \varepsilon < r(u) \).

Proof. By the definition of \( C_1 \) and \( C_2 \) (see Lemma 1 above) we have
\[
\tilde{u}_1(x, \lambda) = C_1(\lambda) u_1(x, \lambda) + C_2(\lambda) u_2(x, \lambda)
\]
(when \( x > p \)). It then follows that

\[
C_1(\lambda) = \frac{W(\tilde{u}_1, u_2)}{W(u_1, u_2)}(\lambda)
\]

\[
C_2(\lambda) = \frac{W(\tilde{u}_1, u_1)}{W(u_2, u_1)}(\lambda)
\]

from which we conclude that \( C_1 \) and \( C_2 \) are analytic if \( \tilde{u}_1, u_1, u_2 \) are.

Suppose now that \( \lambda = u + i\varepsilon, \varepsilon > 0, \) is such that

\[
W(\tilde{u}_1, u_1)(\lambda) \neq 0.
\]

Then,

\[
\left| \frac{C_1(\lambda)}{W(\tilde{u}_1, u_1)(\lambda)} + \frac{m(\lambda)}{W(u_1, u_2)(\lambda)} \right| \geq \left| \text{Im} \frac{m(\lambda)}{W(u_1, u_2)(\lambda)} - \text{Im} \frac{C_1(\lambda)}{W(\tilde{u}_1, u_1)(\lambda)} \right|.
\]

The function \( \rho_1 \) in addition to being absolutely continuous has by hypothesis (a) the property

\[
\frac{d\rho_1}{d\lambda} \bigg|_{\lambda = I} \geq N > 0.
\]

This implies that for \( u \in I', 0 < \varepsilon < k', k' \) small enough,

\[
\int_{-\infty}^{\infty} \frac{\varepsilon}{(\mu - u)^2 + \varepsilon^2} \, d\rho_1(\mu) \geq K > 0
\]

and we conclude that there is a constant \( k'' > 0 \) such that if \( 0 < \varepsilon < k'' \) then

\[
-\text{Im} \frac{m}{W(u_1, u_2)}(u + i\varepsilon) \geq K > 0
\]

for \( u \in I' \). (See [2].)

Let us note that \( C_1(\lambda) \) and \( W(\tilde{u}_1, u_1)(\lambda) \) are real whenever \( \lambda \) is real. Therefore

\[
\left| \text{Im} \frac{m(u + i\varepsilon)}{W(u_1, u_2)(u + i\varepsilon)} - \text{Im} \frac{C_1(u + i\varepsilon)}{W(\tilde{u}_1, u_1)(u + i\varepsilon)} \right| \geq N(u) > 0
\]

if \( \varepsilon < k(u), u \in I', k \) small enough.

Now, we are assuming that \( \lambda = u + i\varepsilon, \varepsilon > 0, \) is such that \( W(\tilde{u}_1, u_1)(\lambda) \neq 0 \), therefore

\[
|W(\tilde{u}_1, u_1)(\lambda)| \left| \frac{C_1(\lambda)}{W(\tilde{u}_1, u_1)(\lambda)} + \frac{m(\lambda)}{W(u_1, u_2)(\lambda)} \right| \geq |W(\tilde{u}_1, u_1)(\lambda)|N(u) > 0
\]

if \( 0 < \varepsilon < k(u) \).
If we recall the form of $C_2$ we obtain

$$|C_1(\hat{\lambda}) - C_2(\hat{\lambda}) m(\hat{\lambda})| \geq |W(\hat{u}_1, u_1)(\hat{\lambda})| N(u) \quad \text{if} \quad 0 < \varepsilon < k(u).$$

Since $W(\hat{u}_1, u_1)(u) \neq 0$ because $u \in I'$ we can take a closed ball $B_u(r)$ with center in $u$ and radius $r$ small enough, in particular $r < k(u)$, such that $W(\hat{u}_1, u_1)(\hat{\lambda}) \neq 0$ for every $\hat{\lambda} \in B_u(r)$.

Since $W(\hat{u}_1, u_1)(\hat{\lambda})$ is analytic it reaches its minimum $M$ in $B_u(r)$. Hence

$$|C_1(u + i\varepsilon) - C_2(u + i\varepsilon) m(u + i\varepsilon)| \geq MN(u) = N(u) > 0$$

if $0 < \varepsilon < r(u)$ for $u \in I'$.

With the help of the preceding lemmas we shall prove the following result. Remember that $\hat{m}$ denotes the WTK coefficient of $\hat{L}$ with respect to $\hat{u}_1(x, z), \hat{u}_2(x, z)$.

**Lemma 4.** *If the operators $L_\alpha$ and $L_\beta$ do not have exactly the same spectrum then it is not possible that*

$$\lim_{\varepsilon \downarrow 0} |\hat{m}(u + i\varepsilon)| = \infty$$

*for* $u \in I' = I \setminus A$.

**Proof.** Suppose we have

$$\lim_{\varepsilon \downarrow 0} |\hat{m}(u + i\varepsilon)| = \infty.$$

Using Lemma 1 we have

$$\lim_{\varepsilon \downarrow 0} \left| \frac{m(\hat{\lambda})}{C_1(\hat{\lambda}) - C_2(\hat{\lambda}) m(\hat{\lambda})} \right| = \infty,$$

where $\hat{\lambda} = u + i\varepsilon, u \in I'$.

From Lemma 3 we know that we can choose $k(u)$ small enough so that if $0 < \varepsilon < k(u), u \in I'$, then

$$|C_1(u + i\varepsilon) - C_2(u + i\varepsilon) m(u + i\varepsilon)| \geq N(u) > 0.$$

Moreover, since we are supposing that $\lim_{\varepsilon \downarrow 0} |\hat{m}(u + i\varepsilon)| = \infty$, we can choose $k(u)$ small enough so that

$$|\hat{m}(u + i\varepsilon)| > M > 0 \quad \text{if} \quad 0 < \varepsilon < k(u)$$

holds, where $M > 0$ is a given arbitrary constant.
Therefore if $0 < \varepsilon < k(u)$ then

$$|m(u + i\varepsilon)| \geq N(u) M > 0.$$  

Since $M$ is arbitrary it follows that

$$\lim_{\varepsilon \downarrow 0} |m(u + i\varepsilon)| = \infty \quad \text{for} \quad u \in I'.$$

Now, if $|m(\lambda)| \neq 0$ we have

$$|\tilde{m}(\lambda)| = \frac{1}{\left| \frac{C_1(\lambda)}{|m(\lambda)|} - \frac{C_2(\lambda) m(\lambda)}{|m(\lambda)|} \right|} \leq \frac{1}{|C_2(\lambda)| - \left| \frac{C_1(\lambda)}{m(\lambda)} \right|}$$

therefore

$$\lim_{\varepsilon \downarrow 0} |\tilde{m}(u + i\varepsilon)| \leq \lim_{\varepsilon \downarrow 0} \frac{1}{\left| \frac{C_2(u + i\varepsilon)}{|m(u + i\varepsilon)|} - \frac{C_1(u + i\varepsilon)}{m(u + i\varepsilon)} \right|} = \frac{1}{|C_2(u)|} = \left| \frac{W(u_2, u_1)}{W(\tilde{u}_1, u_1)} (u) \right| < \infty$$

since if $u \in I'$ we have $W(\tilde{u}_1, u_1) \neq 0$. But this contradicts the assumption

$$\lim_{\varepsilon \downarrow 0} |\tilde{m}(u + i\varepsilon)| = \infty. \quad \text{Q.E.D.}$$

4. Proof of the Theorem

Let $\tilde{\rho}$ be the spectral function of the operator $\tilde{L}$. We shall prove first that $\tilde{\rho}$ is continuous in $I$.

Let $\theta L$ be the selfadjoint operator generated by the differential expression

$$(lu)(x) = -u''(x) + q(x) u(x), \quad 0 \leq x < \infty$$

and the boundary condition

$$u(0) \cos \theta + u'(0) \sin \theta = 0, \quad \theta \in [0, 2\pi).$$

(In particular $\chi L = L$.)
**Lemma 5.** The operators $\theta L$ have only absolutely continuous spectrum in $I$ if $\theta \neq x$.

For the proof of this lemma see [3].

**Lemma 6.** The spectral function $\tilde{\rho}$ of the operator $\tilde{L}$ is continuous in $I$.

**Proof.** We shall prove that

\[ \tilde{T}u = \lambda u, \quad \lambda \in I \]

(2)

does not have $L_2$ solutions.

To prove this suppose that $v$ is a solution of (2) and that $v$ belongs to $L_2(0, \infty)$. Then we have that

\[ (\tilde{T}v)(x) = (lv)(x) = \lambda v(x) \]

if $x \geq p$ and $v \in L_2(p, \infty)$.

Let $\{v_1, v_2\}$ be a system of solution of $hv_k = zv_k$ for $x \in [0, \infty)$ which satisfy

\[
\begin{align*}
v_1(p, z) &= 1, & v_2(p, z) &= 0 \\
v'_1(p, z) &= 0, & v'_2(p, z) &= 1
\end{align*}
\]

for all $z \in \mathbb{C}$.

There exist $C_1(\lambda), C_2(\lambda) \in \mathbb{C}$ such that

\[ h(x, \lambda) := C_1(\lambda) v_1(x, \lambda) + C_2(\lambda) v_2(x, \lambda) = v(x, \lambda) \quad \text{for} \quad x \in [p, \infty). \]

It follows that $h(x) \in L_2(0, \infty)$ and that

\[ (lh)(x) = \lambda h(x). \]

But Lemma 5 implies that $lu = \lambda u$ does not have solutions in $L_2(0, \infty)$ when $\lambda \in I$. Therefore we have a contradiction and (2) does not have $L_2$ solutions. This implies that the spectral function $\tilde{\rho}$ is continuous (see [4]).

Q.E.D.

The fact that (2) does not have $L_2$ solutions implies (see [4]) that $I$ is contained in the continuous spectrum of any selfadjoint realization of $\tilde{T}$ in $L_2(0, \infty)$.

Now let us define the symmetric derivative of the spectral function of the operator $\tilde{L}$

\[
D\tilde{\rho}(u) := \lim_{\eta \to 0} \frac{\tilde{\rho}(u + \eta) - \tilde{\rho}(u - \eta)}{2\eta}.
\]
LEMMA 7. Suppose that $L_\beta$ and $L_{\beta'}$ do not have exactly the same spectrum. If $D\hat{\rho}(u)$ exists, then

$$|D\hat{\rho}(u)| < \infty \quad \text{for} \quad u \in I' = I \setminus A.$$ 

Proof. We know that

$$-\text{Im} \frac{\tilde{m}(u + i\epsilon)}{W(\tilde{u}_1, \tilde{u}_2)(u + i\epsilon)} - \text{Im} \ H(u + i\epsilon) = \int_{-\infty}^{\infty} \frac{\epsilon}{(u - \mu)^2 + \epsilon^2} d\hat{\rho}(\mu),$$

where $H$ is an analytic function. $H(z)$ is real if $z$ real. See [2].

Furthermore we have that

$$\int_{-\infty}^{\infty} \frac{\epsilon}{(u - \mu)^2 + \epsilon^2} d\hat{\rho}(\mu) \geq \frac{1}{\epsilon} \int_{u - \epsilon}^{u + \epsilon} \frac{\epsilon^2}{(u - \mu)^2 + \epsilon^2} d\hat{\rho}(\mu) \geq \frac{\hat{\rho}(u + \epsilon) - \hat{\rho}(u - \epsilon)}{2\epsilon}.$$ 

Therefore

$$\left| \frac{\tilde{m}(u + i\epsilon)}{W(\tilde{u}_1, \tilde{u}_2)(u + i\epsilon)} \right| - \text{Im} \ H(u + i\epsilon) \geq \frac{\hat{\rho}(u + \epsilon) - \hat{\rho}(u - \epsilon)}{2\epsilon} \geq 0.$$ 

From this inequality we see that if $D\hat{\rho}(u) = \infty$ then

$$\lim_{\epsilon \downarrow 0} |\tilde{m}(u + i\epsilon)| = \infty \quad \text{for} \quad u \in I',$$

which contradicts Lemma 4. Q.E.D.

If $\hat{\rho}'(\lambda) := \lim_{h \to 0} ((\hat{\rho}(\lambda + h) - \hat{\rho}(\lambda))/h)$ exists, then $D\hat{\rho}(\lambda)$ exists and $D\hat{\rho}(\lambda) = \hat{\rho}'(\lambda).$ Therefore by Lemma 7 above $\hat{\rho}'(\lambda) < \infty$ if $\lambda \in I'.$

Proof of the Theorem. Let $E_\infty \subset I$ be the set of points $\lambda$ where $\hat{\rho}'(\lambda) = \infty.$ Let us denote also by $\hat{\rho}$ the measure associated with the monotone non-decreasing function $\hat{\rho}.$ By a theorem of de la Vallée-Poussin (see [8]) we know that if $X \subset I$ is a measurable set then

$$\hat{\rho}(X) = \hat{\rho}(X \cap E_\infty) + \int_X \hat{\rho}'(x) \, dx.$$ 

Since $\hat{\rho}$ is continuous by Lemma 6 and $E_\infty \subset A,$ where $A$ is a finite set by Lemma 7, it follows that $\hat{\rho}$ is absolutely continuous in $I$ and the theorem is proved. Q.E.D.

The theorem requires $L_\beta$ and $L_{\beta'}$ not to have exactly the same spectrum. Now we shall construct $\nu: \mathbb{R}^+ \to \mathbb{R}$ continuous and with compact support $S \subset \mathbb{R}$ such that for every $\beta \in [0, 2\pi)$ this condition is satisfied.

We know that there exists an increasing unbounded sequence
\(\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots\) of eigenvalues of \(L_x\) and that the eigenfunction corresponding to the \(n\)-th eigenvalue has exactly \(n\) zeros in the open interval \((0, p)\).

(See, for example, [6].)

Choose \(\lambda_n\) such that \(n \geq 4\) and

\[
\lambda_n > M = \sup_{x \in (0, p)} |q(x)|.
\]

Choose an open interval \(J = (a, b) \subset (0, p)\) such that the \(n\) zeros of the eigenfunction corresponding to \(\lambda_n\) are contained in \(J\) and let \(k > \lambda_n\).

Let \(v : \mathbb{R}^+ \to \mathbb{R}\) be a continuous function satisfying

(a) \(v(x) = k - q(x)\) if \(x \in J\),

(b) \(v(x) > 0\) if \(x \in (0, p)\),

(c) \(v(x) = 0\) if \(x \in \mathbb{R}^+ \setminus (0, p)\)

(in particular \(v(0) = v(p) = 0\)).

Define then

\[\tilde{q}(x) := q(x) + v(x)\]

Using the comparison theorem (see [1, 6]) we can prove that the solutions of

\[-u'' + (\tilde{q}(x) - \lambda_n)u = 0\]

have at most three zeros in \((0, p)\), proving that the \(n\)-th eigenvalue of \(L_x\) cannot be the \(n\)-th eigenvalue of \(L_x\).

Therefore we have constructed a perturbation which satisfies the hypothesis of the theorem for every \(\beta \in [0, 2\pi)\).

Now let \(L\) be the operator

\[Lu \equiv -u'' + qu\]

subject to the boundary conditions

\[u(0) \cos \alpha + u'(0) \sin \alpha = 0\]

\[u(1) \cos \beta + u'(1) \sin \beta = 0, \quad \alpha, \beta \in [0, \pi)\]

In Hochstadt and Lieberman [5] the following result is proved:

**Theorem.** (See [5]). Consider the operator (3) subject to (4) and (5) where \(q\) is summable on \((0, 1)\). Let \(\{\lambda_n\}\) be the spectrum of \(L\) subject to (4) and (5).

Consider a second operator

\[\tilde{L}u \equiv -u'' + \tilde{q}u,
\]
where $\bar{q}$ is summable on $(0, 1)$ and

$$\bar{q}(x) = q(x) \quad \text{on} \quad \left(\frac{1}{2}, 1\right).$$

Suppose that the spectrum of $\bar{L}$ subject to (4) and (5) is also $\{\lambda_n\}$. Then $q(x) = \bar{q}(x)$ almost everywhere on $(0, 1)$.

By scaling we can take $(0, p)$ instead of $(0, 1)$ and $\bar{q}(x) = q(x)$ on $(\frac{1}{2}p, p)$ instead of $\bar{q}(x) = q(x)$ on $(\frac{1}{2}, 1)$.

With the help of this theorem if we choose $p \in \mathbb{R}$ such that $S \subset (0, \frac{1}{2}p)$, where $S$ is the support of the perturbation $v(x)$, then it follows that $L_\alpha$ and $L_\beta$ do not have the same spectrum when $\alpha = \beta$, unless $v(x) \equiv 0$. Then by the theorem proved in this work it follows that every perturbation $v(x)$ continuous and not identically null changes singular continuous spectrum into absolutely continuous spectrum.

We have proved therefore the following result, where $I \subset \mathbb{R}$ is an interval.

**COROLLARY.** There exists a continuous potential $q(x)$ and $\alpha \in [0, \pi)$ such that

$$lu = -u'' + q(x)u, \quad x \in [0, \infty)$$

$$u(0) \cos \alpha + u'(0) \sin \alpha = 0$$

has singular continuous spectrum in $I$ and such that for every continuous function $v(x)$, not identically null and with compact support, the operator generated by

$$\bar{L}u = -u'' + \{q(x) + v(x)\}u, \quad x \in [0, \infty)$$

and the boundary condition

$$u(0) \cos \alpha + u'(0) \sin \alpha = 0$$

has only absolutely continuous spectrum in $I$.

**REFERENCES**


