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ON BOUNDARY CONDITIONS OF AN INVERSE STURM-LIOUVILLE PROBLEM*

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Abstract. It is known that if the potential function $q(x)$ in a Sturm-Liouville problem is prescribed over the interval $(0, 1/2)$ and the boundary condition at the point zero is fixed, then a single spectrum suffices to determine $q(x)$ on the interval $(1/2, 1)$ uniquely.

Explicit examples are given that show the necessity of specifying the boundary condition at zero even if the boundary condition at one is given.

Key words. inverse Sturm-Liouville problem, self-adjoint operator, isospectral potentials, eigenvalues

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1. Introduction.

Consider the eigenvalue problem

$$-u''+q(x)u=\lambda u \qquad 0 \leqq x \leqq 1 \tag{1}$$

$$\begin{aligned} &u(0)\cos\alpha+u'(0)\sin\alpha=0\\ &u(1)\cos\beta+u'(1)\sin\beta=0. \end{aligned} \tag{2}$$

Here λ is a complex parameter, q is a real-valued function which is integrable on $[0, 1]$, and α and β are points in $[0, \pi)$.

By definition, a function y defined on $[0, 1]$ is a solution of (1) if it is continuously differentiable, y' is absolutely continuous and the equation holds almost everywhere. If, in addition, y satisfies the conditions (2) then it is said that y solves the problem (1), (2).

It is well known that there exists a sequence of real numbers $\{\lambda_n(\alpha, \beta, q)\}_{n=0}^{\infty}$ called eigenvalues, such that the problem (1) and (2) can be solved.

In 1978 Hochstadt and Lieberman [2] proved the following theorem.

THEOREM 1 (Hochstadt and Lieberman). *Suppose that*

(a) $\lambda_n(\alpha, \beta, q)=\lambda_n(\tilde{\lambda}, \tilde{\beta}, \tilde{q})$ for $n=0, 1, \cdots$,

(b) $q(x)=\tilde{q}(x)$ on $(0, 1/2)$,

(c) $\alpha=\tilde{\alpha}$,

(d) $\beta=\tilde{\beta}$,

then $q(x)=\tilde{q}(x)$ *almost everywhere on* $(0, 1)$.

In [2] the interval $(1/2, 1)$ was considered instead of $(0, 1/2)$.

Shortly after the publication of this theorem, Hald [1] showed that it is possible to ignore condition (d) and still obtain the assertion of the theorem. The related question of whether it is possible to alter the conclusion of the theorem by neglecting condition (c) is the subject of this work.

We show the necessity of condition (c) by constructing two different potentials q and $\tilde{q}$ such that

(i) $\lambda_n(\alpha, \beta, q)=\lambda_n(\tilde{\alpha}, \tilde{\beta}, \tilde{q})$ for $n=0, 1, \cdots$ with $\alpha \neq \tilde{\alpha}$ and $\beta=\tilde{\beta}=0$,

(ii) $q(x)=\tilde{q}(x)$ for $x \in [0, p)$,

where p is an arbitrary point in $(0, 1)$ and thus answer the above question affirmatively.

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2. The result. Let p be an arbitrary but fixed point in the open interval $(0, 1)$. For $x \in [p, 1]$ consider the problem

$$-y''(x) + v(x)y(x) = \mu y(x) \tag{3}$$

with boundary conditions

$$y(p) - y'(p) = 0$$
$$y(1) = 0$$

where $v(x)$ is an arbitrary integrable function.

Let $\{\mu_i\}_{i=0}^{\infty}$ be the eigenvalues and $\{\phi_i\}_{i=0}^{\infty}$ the corresponding eigenfunctions.

Now consider the potential q defined for $x \in [0, 1]$ as follows:

$$q(x) = \begin{cases} 1 + \mu_0 & \text{if } x \in [0, p] \\ v(x) & \text{if } x \in (p, 1] \end{cases}$$

and the problem

$$-y''(x) + q(x)y(x) = \lambda y(x) \qquad x \in [0, 1] \tag{4}$$

with boundary conditions

$$\begin{aligned} y(0) - y'(0) &= 0 \\ y(1) &= 0. \end{aligned} \tag{5}$$

Let us define

$$g_0(x) = \begin{cases} \sqrt{2}\, e^x & \text{for } x \in [0, p] \\ \phi_0(x) & \text{for } x \in [p, 1] \end{cases}$$

where we have normalized ϕ_0 in such a way that $\phi_0(p) = \sqrt{2}\, e^p$ holds. The function g_0 is continuously differentiable, solves (4), (5) for $\lambda = \mu_0$, and since μ_0 is the first eigenvalue of (3), g_0 does not have any roots in $[0, 1)$. Define

$$q_1(x) := q(x) - 2 \frac{d^2}{dx^2} \log \left(1 + \int_0^x g_0^2(s)\, ds \right).$$

It is easy to see that q_1 is integrable on $[0, 1]$.

We now state our main result.

THEOREM 2. *The problems*

$$-y'' + q_1(x)y = \lambda y \qquad x \in [0, 1]$$
$$y(0) + y'(0) = 0$$
$$y(1) = 0$$

and

$$-y'' + q(x)y = \lambda y \qquad x \in [0, 1]$$
$$y(0) - y'(0) = 0$$
$$y(1) = 0$$

have the same spectrum $\{\lambda_i\}_{i=1}^{\infty}$. *In addition to this*

$$q_1(x) = q(x) \quad \text{for} \quad x \in [0, p] \text{ and } q_1 \neq q.$$

Since q_1 and q are integrable functions, the equations are understood to hold almost everywhere.

Proof of Theorem 2. From Lemma 1 in the Appendix we see that

$$h(x) := \frac{1}{g_0(x)}\left(1+\int_0^x g_0^2(s)\,ds\right)$$

is the solution of

$$-y''+\left(q-2\frac{d}{dx^2}\log g_0\right)y=\mu_0 y.$$

Since $g_0(1)=0$, the function h and the potential $q-2(d^2/dx^2)\log g_0$ are defined only in $[0, 1)$ and it is in this interval where the above equation makes sense.

Now let $\{g_j\}_{j=0}^{\infty}$ be the set of eigenfunctions of (4), (5) and let $\{\lambda_i\}$ be the corresponding eigenvalues. Note that $\mu_0=\lambda_0$.

For $j=0$ the corresponding eigenfunction is the g_0 defined above (remember that g_0 and ϕ_0 have the same number of roots).

To begin with, assume $j>0$. For this case we apply Lemma 2 of the Appendix with

$$g=g_0, \qquad f=g_j$$

and h defined as above, exactly as it is done in the proof of Theorem 1 of [3, p. 92].

In this way we obtain that

$$\bar{g}_{jc}=g_j-\frac{1}{g_0}\frac{[g_0, g_j]}{\lambda_0-\lambda_j}\frac{d}{dx}\log\theta_{0c}, \tag{6}$$

with

$$\theta_{0c}=1+c\int_0^x g_0^2(s)\,ds,$$

is a solution of

$$-y''+q_c y=\lambda_j y \quad \text{when } j>0, \tag{7}$$

with

$$q_c=q-2\frac{d^2}{dx^2}\log\theta_{0c},$$

where $[\cdot,\cdot]$ denotes the Wronskian.

It should be noted that since $g_0(1)=0$, the function $\bar{g}_{jc}$ is not defined at the endpoint 1. The potential q_c is defined for every point on $[0, 1]$ and it is integrable when $c>-1$. This follows from the fact that θ_{0c} is strictly positive when $c>-1$.

We extend now the definition of $\bar{g}_{jc}$ to the whole interval $[0, 1]$ including the point 1.

Using the equality

$$[g_0, g_j]=-(\lambda_0-\lambda_j)\int_x^1 g_0(s)g_j(s)\,ds$$

in (6) we obtain

$$\bar{g}_{jc}=g_j+\frac{1}{g_0}\int_x^1 g_0(s)g_j(s)\,ds\cdot\frac{d}{dx}\log\theta_{0c}. \tag{8}$$

Now, using l'Hôpital's rule we obtain that

$$\lim_{x\to 1} \bar{g}_{jc}(x) = g_j(1) = 0.$$

Therefore we define g_{jc} on the whole interval $[0, 1]$ by setting

$$g_{jc}(x) = \begin{cases} \bar{g}_{jc}(x) & \text{if } x \in [0, 1) \\ 0 & \text{if } x = 1. \end{cases}$$

The function g_{jc} is continuous on $[0, 1]$.

We can also extend continuously $\bar{g}'_{jc}$ if we observe that

$$\lim_{x\to 1} \bar{g}'_{jc}(x) = g'_j(1)$$

and define

$$g'_{jc}(x) = \begin{cases} \bar{g}'_{jc}(x) & \text{if } x \in [0, 1) \\ g'_j(1) & \text{if } x = 1. \end{cases}$$

Therefore, the function g_{jc} is continuously differentiable on $[0, 1]$.

Since $\bar{g}_{jc}$ satisfies the equality

$$-\bar{g}''_{jc} = (\lambda_j - q_c)\bar{g}_{jc} \quad \text{a.e.,}$$

then g_{jc} satisfies also

$$-g''_{jc} = (\lambda_j - q_c)g_{jc} \quad \text{a.e.}$$

Now since for $c > -1$ the member on the right of the equality is integrable, it follows that g_{jc} is a genuine solution of (7) on the interval $[0, 1]$.

Now let us see which boundary conditions satisfies g_{jc} at 0 with $c = 1$.

If we consider $x \in [0, p]$, instead of g_0 we can write $\sqrt{2}\, e^x$ and therefore we have

$$\frac{d}{dx} \log \theta_{01} = \frac{[1 + \int_0^x 2\, e^{2s}\, ds]'}{1 + \int_0^x 2\, e^{2s}\, ds} = \frac{2\, e^{2x}}{1 + e^{2x} - 1} = 2$$

when $x \in [0, p]$.

Hence (8) becomes

$$g_{j1}(x) = g_j(x) + \frac{2}{g_0(x)} \int_x^1 g_0(s) g_j(s)\, ds \tag{9}$$

when $x \in [0, p]$.

Now taking the derivative in (9) we obtain

$$g'_{j1}(x) = g'_j(x) - 2g_j(x) - 2\frac{g'_0(x)}{g_0^2(x)} \int_x^1 g_0(s) g_j(s)\, ds;$$

therefore,

$$g'_{j1}(0) = g'_j(0) - 2g_j(0) = g'_j(0) - g_j(0) - g_j(0) = -g_j(0),$$

i.e.,

$$g'_{j1}(0) + g_j(0) = 0.$$

Since $g_{j1}(0) = g_j(0)$ we have

$$g'_{j1}(0) + g_{j1}(0) = 0.$$

Therefore, we conclude that g_{j1} is a solution of the problem

$$\begin{aligned} -y''+q_1(x)y=\lambda_j y \qquad x\in[0,1] \qquad j>0,\\ y(0)+y'(0)=0,\\ y(1)=0, \end{aligned}$$

if g_j is a solution of the problem

$$\begin{aligned} -y''+q(x)y=\lambda_j y \qquad x\in[0,1] \qquad j>0,\\ y(0)-y'(0)=0,\\ y(1)=0. \end{aligned}$$

Now we consider the case $j=0$.

From Lemma 2 of the Appendix it follows that

$$g_{01}(x)=\frac{1}{h}=\frac{g_0(x)}{1+\int_0^x g_0^2(s)\,ds}$$

is a solution of (7), with $j=0$, $c=1$.

In addition to this,

$$g_{01}(1)=0$$

and if we take $x\in[0,p]$, replacing $g_0(x)=\sqrt{2}\,e^x$ we obtain

$$g'_{01}(x)=\left(\frac{\sqrt{2}\,e^x}{1+e^{2x}-1}\right)'=(\sqrt{2}\,e^{-x})'=-\sqrt{2}\,e^{-x}$$

and evaluating at zero, we have

$$g'_{01}(0)=-\sqrt{2}=-g_0(0).$$

Since we have also

$$g_{01}(0)=g_0(0)$$

it follows that

$$g_{01}(0)+g'_{01}(0)=0.$$

Therefore when $j=0$ we have that $g_{01}=1/h$ is a solution of the problem

$$-y''+q_1(y)=\lambda_0 y$$

with boundary conditions

$$\begin{aligned} y(0)+y'(0)=0,\\ y(1)=0. \end{aligned}$$

Now for every real number c, $g_{jc}(0)=g_j(0)$ and $g_{jc}(1)=g_j(1)$. Moreover, g_{jc} solves

$$-y''+q_c y=\lambda_j y$$

and therefore its roots are simple; that is, if $g_{jc}(x_0)=0$, then $g'_{jc}(x_0)\neq 0$. Besides, g_{jc} is a continuous differentiable function of c and x and it has only a finite number of roots in $[0,1]$. Thus, from Lemma 3 of [3, p. 41] it follows that g_{j1} and g_j have the same number of roots in $[0,1]$.

Hence the problems

$$\begin{aligned} -y'' + q_1(x)y &= \lambda y \qquad x \in [0, 1], \\ y(0) + y'(0) &= 0, \\ y(1) &= 0, \end{aligned}$$

and

$$\begin{aligned} -y'' + q(x)y &= \lambda y \qquad x \in [0, 1] \\ y(0) - y'(0) &= 0, \\ y(1) &= 0, \end{aligned}$$

have the same spectrum $\{\lambda_i\}_{i=1}^{\infty}$.

Besides, since

$$\frac{d^2}{dx^2} \log \theta_{01} = \frac{d}{dx}(2) = 0$$

on $[0, p]$, we have

$$q_1(x) = q(x)$$

in the same interval.

In order to prove that $q_1 \not\equiv q$ let us assume that $q_1(x) = q(x)$ almost everywhere in $[p, 1]$. Hence we have

$$\begin{aligned} q_1(x) - q(x) &= -2 \frac{d^2}{dx^2} \log \left[1 + \int_0^x g_n^2(s)\, ds \right] \\ &= 0 \quad \text{a.e.} \end{aligned}$$

Now

$$\frac{d^2}{dx^2} \log \left[1 + \int_0^x g_n^2(s)\, ds \right] = 0$$

implies

$$\frac{[1 + \int_0^x g_n^2(s)\, ds]'}{1 + \int_0^x g_n^2(s)\, ds} = k$$

where k is a constant.

This implies

$$g_n^2(x) = C\, e^{kx} \quad \text{for} \quad x \in [p, 1]$$

and from here we obtain a contradiction. Therefore, the proof of the theorem is complete.

3. Conclusion. From the examples constructed above it follows that condition (c) of Theorem 1 cannot be neglected. Therefore, even if the boundary condition at one is given, it is necessary to specify the boundary condition at zero.

4. Appendix. The proof of the following lemmas can be found in Pöschel and Trubowitz [3]. The symbol $[\cdot, \cdot]$ denotes the Wronskian.

LEMMA 1. *Pick a real number μ, and let g be a nontrivial solution of*

$$-y'' + qy = \lambda y \tag{10}$$

for $\lambda = \mu$. If f is a nontrivial solution of (10) *for $\lambda \neq \mu$, then*

$$\frac{1}{g}[g, f]$$

is a nontrivial solution of

$$-y'' + \left(q - 2\frac{d^2}{dx^2}\log g\right)y = \lambda y \tag{11}$$

for the same λ. Also, for $\lambda = \mu$, the general solution of (11) *is given by*

$$\frac{1}{g}\left(a + b\int_0^x g^2(s)\,ds\right),$$

where a and b are arbitrary constants. In particular, $1/g$ is a solution.

If g has roots in $[0, 1]$ *then* (11) *is understood to hold between them.*

LEMMA 2. *Pick real numbers μ and ν. Let g be a nontrivial solution of* (10) *for $\lambda = \mu$ and h a nontrivial solution of* (11) *for $\lambda = \nu$. If f is a nontrivial solution of* (10) *for $\lambda \neq \mu$, ν then*

$$\frac{1}{h}\left[h, \frac{1}{g}[g, f]\right] = (\mu - \lambda)f - \frac{1}{g}[g, f]\frac{d}{dx}\log(gh)$$

is a nontrivial solution of

$$-y'' + \left(q - 2\frac{d^2}{dx}\log(gh)\right)y = \lambda y \tag{12}$$

for the same λ. Also, $1/h$ is a nontrivial solution of (12) *for $\lambda = \nu$.*

Equation (12) *is understood to hold between the roots of g and h.*

REFERENCES

[1] O. H. HALD, *Inverse eigenvalue problems for the mantle*, Geophys. J. R. Astrophys. Soc., 62 (1980), pp. 41-48.

[2] H. HOCHSTADT AND B. LIEBERMAN, *An inverse Sturm-Liouville problem with mixed given data*, SIAM J. Appl. Math., 34 (1978), pp. 676-680.

[3] J. PÖSCHEL AND E. TRUBOWITZ, *Inverse Spectral Theory*, Academic Press, New York, 1987.