ON CONDITIONS WHICH GUARANTEE STABILITY OF THE ABSOLUTELY CONTINUOUS SPECTRUM OF ORDINARY DIFFERENTIAL OPERATORS

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1. Introduction

IN **[11]** J. Weidmann proved the absolute continuity of the spectrum of Sturm–Liouville operators under very general conditions. The following differential expressions were considered

$$(lu)(x) = -u''(x) + q(x)u(x) \quad \text{for} \quad x \in (a, \infty),$$

where q is a real valued, locally integrable function defined in (a, ∞).

In **[11]**, aside from conditions on q in the interval (c, ∞) with $a < c < \infty$, there were (implicit) conditions on the behavior of q by the point a: to prove the absolute continuity of the spectrum of all the self-adjoint realizations of l in an interval $(\underline{\lambda}, \bar{\lambda})$ it was necessary that all self-adjoint realizations of l in $L_2(a, c)$ had only discrete spectrum in $(\underline{\lambda}, \bar{\lambda})$.

The same result was proved without this additional condition by a in E. Heinz **[5]** for Dirac Systems and in J. Weidmann **[12]** for Dirac Systems and Sturm–Liouville operators. In **[5]** the behavior of the resolvent near the real axis was investigated whereas in **[11]** and also in **[12]** oscillation methods were used.

The fact that the absolute continuity of the spectrum in these cases cannot be perturbed by the behavior at the left point a, suggests the following conjecture (which goes back essentially to J. Weidmann **[12]**): Let l be a formally self-adjoint differential expression in (a, b) and A a self-adjoint realization. For $c \in (a, b)$ let each self-adjoint realization A_b of l in (c, b) have absolutely continuous spectrum in $(\underline{\lambda}, \bar{\lambda})$. Then A also has absolutely continuous spectrum in $(\underline{\lambda}, \bar{\lambda})$. (We say that a self-adjoint operator has absolutely continuous spectrum in an interval I when $\|E(\lambda)f\|^2$ is absolutely continuous in I with respect to the Lebesgue measure for all $f \in L_2(a, b)$, $E(\lambda)$ being the family of spectral projections of the operator).

It was proved in **[3]**, using the theory of Weyl–Titchmarsh–Kodaira and the theorem of Gelfand–Levitan, that the absolute continuity of all the self-adjoint realizations A_b is not enough to imply the absolute continuity of A and so the above conjecture was refuted.

In this paper we shall prove that it is enough to give conditions on two self-adjoint realizations of l in (c, b) in order to have absolute continuity for every self-adjoint realization of l in (a, b). We then apply this result to Sturm–Liouville expressions with coefficients periodic in (c, ∞) and arbitrary (only locally integrable) in (a, c). Corollary II.6 of Carmona [**1**] is a particular case of this application. We shall also give a generalization of a result of Hinton–Shaw for perturbed periodic potentials.

The above results can be extended to Dirac Systems. As an application we prove that the absolutely continuous spectrum of the Dirac Systems investigated by Hinton–Shaw [**7**] is independent of the behavior of the coefficients by the left point a.

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2. Main result

Let l be the following differential expression

$$lu = -u'' + q(x)u \qquad a < x < b \qquad 0 \in (a, b) \tag{1}$$

(it is possible to have $a = -\infty$ or $b = \infty$) where q is a real-valued, locally integrable function defined in (a, b). We will also assume that the limit point case ($l.p.c.$) occurs at b.

Define the operators L_θ and L as follows:

$$\begin{aligned} D(L_\theta) = \{u \in L_2(0, b) \mid & \; u, u' \text{ absolutely continuous} \\ & \text{in } (0, b),\, lu \in L_2(0, b)\; u(0)\cos\theta + u'(0)\sin\theta = 0\} \end{aligned}$$

(since the endpoint 0 is regular we have the limit circle case ($l.c.c.$) there)

$$\begin{aligned} L_\theta u &= lu \\ D(L) = \{u \in L_2(a, b) \mid & \; u, u' \text{ absolutely continuous} \\ & \text{in } (a, b),\, lu \in L_2(a, b)\; [u, v]_a = 0 \text{ when } l.c.c. \text{ at } a\} \end{aligned}$$

(v is a nontrivial real solution of $lv = 0$)

$$Lu = lu.$$

The operators L_θ and L are self-adjoint in $L_2(0, b)$ and $L_2(a, b)$, respectively.

We know (see [**14**] p. 257) that for $z \in \mathbf{C}\backslash\mathbf{R}$ the Resolvent of L has the form

$$\begin{aligned} R_z(L)g(x) &= (z - L)^{-1}g(x) \\ &= W(u_a, u_b)^{-1}\left\{u_b(x)\int_a^x u_a(y)g(y)\,dy + u_a(x)\int_x^b u_b(y)g(y)\,dy\right\} \end{aligned}$$

where $W(.\,,.)$ is the Wronskian and u_a, u_b are solutions of $lu = zu$, uniquely determined up to a constant factor by the conditions: u_a is in $L_2(a, c)$ if we have *l.p.c.* at a or u_a satisfies the boundary condition at a if *l.c.c.* takes place at a; the function u_b is in $L_2(c, b)$.

Let u_1, u_2 be a fundamental set of solutions of $lu = zu$ satisfying

$$\begin{aligned} u_1(0, z) &= \sin\theta \qquad & u_2(0, z) &= \cos\theta \\ u_1'(0, z) &= -\cos\theta \qquad & u_2'(0, z) &= \sin\theta \\ & & \theta &\in [0, 2\pi). \end{aligned}$$

For $z \in \mathbf{C}\backslash\mathbf{R}$ $W(u_1, u_a)(z) \neq 0$ and $W(u_1, u_b)(z) \neq 0$ since otherwise we would have a self-adjoint problem with a nonreal eigenvalue.

Choose $m_b(z)$ and $m_a(z)$ such that

$$u_b(x, z) = m_b(z)u_1(x, z) + u_2(x, z)$$

and $m_a(z)$ satisfying an analogous equality. Then, we can write

$$\begin{aligned} R_z(L)y(x) = {}&[W(m_a(z)u_1(x, z) + u_2(x, z)), (m_b(z)u_1(x, z) + u_2(x, z))]^{-1} \\ &\cdot \Big\{(m_b(z)u_1(x, z) + u_2(x, z)) \int_a^x (m_a(z)u_1(y, z) \\ &+ u_2(y, z))g(y)\, dy + (m_a(z)u_1(x, z) \\ &+ u_2(x, z)) \int_x^b (m_b(z)u_1(y, z) + u_2(y, z))g(y)\, dy\Big\} \end{aligned}$$

This integral operator has the kernel

$$R(x, y, z) = \begin{cases} \sum_{j,k=1}^{2} m_{jk}^{+}(z)u_j(x, z)u_k(y, z) & \text{for} \quad y < x \\ \sum_{j,k=1}^{2} m_{jk}^{-}(z)u_j(x, z)u_k(y, z) & \text{for} \quad x < y \end{cases}$$

where

$$\begin{pmatrix} m_{11}^{+}(z) & m_{12}^{+}(z) \\ m_{21}^{+}(z) & m_{22}^{+}(z) \end{pmatrix} = \begin{pmatrix} \dfrac{m_a(z)m_b(z)}{W(u_a, u_b)(z)} & \dfrac{m_b(z)}{W(u_a, u_b)(z)} \\ \dfrac{m_a(z)}{W(u_a, u_b)(z)} & \dfrac{1}{W(u_a, u_b)(z)} \end{pmatrix}$$

$$\begin{pmatrix} m_{11}^{-}(z) & m_{12}^{-}(z) \\ m_{21}^{-}(z) & m_{22}^{-}(z) \end{pmatrix} = \begin{pmatrix} \dfrac{m_a(z)m_b(z)}{W(u_a, u_b)(z)} & \dfrac{m_a(z)}{W(u_a, u_b)(z)} \\ \dfrac{m_b(z)}{W(u_a, u_b)(z)} & \dfrac{1}{W(u_a, u_b)(z)} \end{pmatrix}$$

$$W(u_a, u_b) = [m_a - m_b]W(u_1, u_2)$$

We call the matrices $(m_{ij}^{\pm})$ the characteristic matrices of the operator L corresponding to the system $\{u_1, u_2\}$. If we choose $u_1 = u_a$ we obtain the following characteristic matrices for L_θ:

$$\begin{pmatrix} m_{11}^{+} & m_{12}^{+} \\ m_{21}^{+} & m_{22}^{+} \end{pmatrix} = \begin{pmatrix} \dfrac{m_b}{W(u_1, u_2)} & 0 \\ \dfrac{1}{W(u_1, u_2)} & 0 \end{pmatrix}$$

$$\begin{pmatrix} m_{11}^{-} & m_{12}^{-} \\ m_{21}^{-} & m_{22}^{-} \end{pmatrix} = \begin{pmatrix} \dfrac{m_b}{W(u_1, u_2)} & \dfrac{1}{W(u_1, u_2)} \\ 0 & 0 \end{pmatrix}$$

Let ρ_θ and ρ be the (2×2 matrix valued) spectral functions of L_θ and L.

The operator L is unitarily equivalent to the operator of multiplication by the variable in $L_2(\mathbf{R}, \mathrm{d}\rho)$. The spectral function is given by the formula of Weyl–Titchmarsh–Kodaira

$$\rho_{kj}(\lambda) = -\frac{1}{\pi} \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_{\delta}^{\lambda+\delta} \operatorname{Im} m_{kj}^{\pm}(s + i\varepsilon)\, \mathrm{d}s.$$

The same is true for L_θ and its spectral function.

For the fundamental system $\{u_1, u_2\}$, with $u_1 = u_a$ we have that the spectral function $\rho_\theta = (\rho_\theta)_{ij}$ has only one component different from zero (abusing of the notation we denote this component by ρ_θ) and

$$-\operatorname{Im} m_b(z) = \int_{-\infty}^{\infty} \frac{\varepsilon}{(u-\lambda)^2 + \varepsilon^2}\, \mathrm{d}\rho_\theta(\lambda) \qquad z = u + i\varepsilon. \tag{2}$$

(See [8] p. 136 Theorem 5.2).

THEOREM 1. *If there exists constants $k > 0$, $N > 0$, $M > 0$ such that the inequalities*:

$$|m_b(u + i\varepsilon)| < N$$

and

$$|\operatorname{Im} m_b(u + i\varepsilon)| > M$$

hold for $0 < \varepsilon < k$ and for all $u \in I = [\underline{\lambda}, \bar{\lambda}]$, then $\rho(\lambda)$ is absolutely continuous in I.

Proof. Let

$$a(z) = \operatorname{Re} m_a(z) \qquad c(z) = \operatorname{Re} m_b(z)$$
$$b(z) = \operatorname{Im} m_a(z) \qquad d(z) = \operatorname{Im} m_b(z).$$

$$-\operatorname{Im} m_{22}^{+} = -\operatorname{Im} \frac{1}{m_a(z) - m_b(z)} = \frac{b-d}{(a-c)^2 + (b-d)^2}.$$

It is known that $b(z) > 0$ and $d(z) < 0$ when $\operatorname{Im} z > 0$. (See [**10**]). Therefore, we have

$$0 < \frac{b-d}{(a-c)^2+(b-d)^2} \leqslant -\frac{1}{d} = -\frac{1}{\operatorname{Im} m_b(z)},$$

and hence

$$-\operatorname{Im} \frac{1}{m_a(u+i\varepsilon) - m_b(u+i\varepsilon)} < \frac{1}{|\operatorname{Im} m_b(u+i\varepsilon)|}.$$

Now, we know that for $0 < \varepsilon < k$, $|\operatorname{Im} m_\alpha(u+i\varepsilon)| > M$ holds; hence,

$$-\operatorname{Im} \frac{1}{m_a(u+i\varepsilon) - m_b(u+i\varepsilon)} < \frac{1}{M}$$

for $u \in I$.

Therefore, we have

$$\rho_{22}(\mu) - \rho_{22}(\gamma) = \frac{1}{\pi} \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_{\gamma+\delta}^{\mu+\delta} \left\{ -\operatorname{Im} \frac{1}{m_a(u+i\varepsilon) - m_b(u+i\varepsilon)} \right\} du$$
$$\leqslant \frac{1}{\pi M} \int_\gamma^\mu du,$$

i.e. ρ_{22} is absolutely continuous.

Now consider

$$-\operatorname{Im} m_{11}^{+}(z) = -\operatorname{Im} \frac{m_a m_b}{m_a - m_b}$$
$$= \frac{\operatorname{Im} \dfrac{1}{m_b} - \operatorname{Im} \dfrac{1}{m_a}}{\left(\operatorname{Re} \dfrac{1}{m_b} - \operatorname{Re} \dfrac{1}{m_a}\right)^2 + \left(\operatorname{Im} \dfrac{1}{m_b} - \operatorname{Im} \dfrac{1}{m_a}\right)^2}$$

(>0 for z in $\mathbf{C}$ with $\operatorname{Im} z > 0$)

$$\leqslant \frac{1}{\operatorname{Im} \dfrac{1}{m_b} - \operatorname{Im} \dfrac{1}{m_a}}$$
$$\leqslant -\frac{|m_b|}{\operatorname{Im} m_b} \leqslant \frac{N}{M}.$$

Hence,

$$\rho_{11}(\mu) - \rho_{11}(\gamma) \leqslant \int_\gamma^\mu du \cdot \frac{1}{\pi} \frac{N}{M}$$

and therefore ρ_{11} is absolutely continuous.

One may also see [**8**, page 155] that

$$\left\{\operatorname{Im}\frac{m_a}{m_a-m_b}\right\}^2=\left\{\operatorname{Im}\frac{m_b}{m_a-m_b}\right\}^2\leqslant\left(\operatorname{Im}\frac{1}{m_a-m_b}\right)\left(\operatorname{Im}\frac{m_a m_b}{m_a-m_b}\right)$$

holds. From here the absolute continuity of ρ_{12} and ρ_{21} follows.

The function m_b depends on $\{u_1, u_2\}$ which depend on θ. Let us now forget about the dependence on b and write m_θ instead of m_b. Then we have the following.

LEMMA 1. *Let α and β be two different values of θ, i.e. $\alpha\in[0, 2\pi)$, $\beta\in[0, 2\pi)$ and $\alpha\neq\beta$. Then we have following equality*:

$$m_\beta(z)=\frac{m_\alpha(z)\cot\gamma-1}{m_\alpha(z)+\cot\gamma},\qquad \gamma=\alpha-\beta$$

(*See* [**10**]).

Now we can give conditions on ρ_θ which guarantee the absolute continuity of ρ.

THEOREM 2. *Let ρ_θ be absolutely continuous in $I=(\underline{\lambda},\bar{\lambda})\subset\mathbf{R}$ for $\theta=\alpha$ and $\theta=\beta$ (two different values of θ). Let M, N, $N^\cdot$ be constants such that the inequalities*

$$0<N<\left.\frac{\mathrm{d}\rho_\alpha}{\mathrm{d}\lambda}\right|_{\lambda\in I}<M<\infty$$

and

$$0<N^\cdot<\left.\frac{\mathrm{d}\rho_\beta}{\mathrm{d}\lambda}\right|_{\lambda\in I}$$

hold.

Then ρ is absolutely continuous in I.

Proof. First we prove two claims:

Claim (*a*). There exists $k>0$ such that for $0<\varepsilon<k$ we have

$$\int_{-\infty}^{\infty}\frac{\varepsilon}{(\lambda-u)^2+\varepsilon^2}\,\mathrm{d}\rho_\alpha(\lambda)<C=\text{constant}$$

uniformly for $u\in I'=[\underline{\lambda}',\bar{\lambda}']\subset \mathrm{I}=(\underline{\lambda},\bar{\lambda})$.

Proof. First, we consider the integral over $\bar{I}=[\underline{\lambda},\bar{\lambda}]$

$$\int_{\underline{\lambda}}^{\bar{\lambda}}\frac{\varepsilon}{(\lambda-u)^2+\varepsilon^2}\,\mathrm{d}\rho_\alpha(\lambda)=\int_{\underline{\lambda}}^{\bar{\lambda}}\frac{\varepsilon}{(\lambda-u)^2+\varepsilon^2}\,\rho_\alpha'(\lambda)\,\mathrm{d}\lambda$$
$$\leqslant M\int_{\underline{\lambda}}^{\bar{\lambda}}\frac{\varepsilon}{(\lambda-u)^2+\varepsilon^2}\,\mathrm{d}\lambda\leqslant M\pi.$$

Now let $u \in I'$. Then we have

$$\int_{\mathbf{R}\setminus\bar{\mathbf{I}}} \frac{\varepsilon}{(\lambda-u)^2+\varepsilon^2}\,\mathrm{d}\rho_\alpha(\lambda) \leqslant \varepsilon C_1 \int_{\mathbf{R}\setminus\bar{\mathbf{I}}} \frac{1}{1+\lambda^2}\,\mathrm{d}\rho_\alpha(\lambda) \leqslant \varepsilon C_1 C_2$$

because there exists a constant $C_1 > 0$ such that

$$\frac{1}{(\lambda-u)^2} < \frac{C_1}{1+\lambda^2}$$

for every $u \in I'$, $\lambda \in \mathbf{R}\setminus\bar{I}$ and

$$\int_{-\infty}^{\infty} \frac{1}{1+\lambda^2}\,\mathrm{d}\rho_\alpha(\lambda) = -\mathrm{Im}\, m_\alpha(i) = C_2 < \infty$$

(see formula (2)). This proves our claim.

Claim (b). There exists k such that for $0 < \varepsilon < k$ we have

$$\int_{-\infty}^{\infty} \frac{\varepsilon}{(\lambda-u)^2+\varepsilon^2}\,\mathrm{d}\rho_\alpha(\lambda) \geqslant D = \text{constant} > 0$$

uniformly for $u \in I'$.

Proof. We consider first the integral over I'

$$\int_{I'} \frac{\varepsilon}{(\lambda-u)^2+\varepsilon^2}\,\mathrm{d}\rho_\alpha(\lambda) = \int_{I'} \frac{\varepsilon}{(\lambda-u)^2+\varepsilon^2}\,\rho'_\alpha(\lambda)\,\mathrm{d}\lambda$$
$$\geqslant N \int_{(\lambda'-u)/\varepsilon}^{(\bar{\lambda}'-u)/\varepsilon} \frac{\mathrm{d}v}{v^2+1} \geqslant NC = D.$$

$(C > 0$ when $\varepsilon < k)$.

Above we have made the change of variables $v = (\lambda - u)/\varepsilon$. Also, since ρ_α does not decrease we have

$$\int_{\mathbf{R}\setminus\mathbf{I}'} \frac{\varepsilon}{(\lambda-u)^2+\varepsilon^2}\,\mathrm{d}\rho_\alpha(\lambda) \geqslant 0.$$

From this inequality claim (b) follows.

Using formula (2) we can write

$$D \leqslant |-\mathrm{Im}\, m_\alpha(u+i\varepsilon)| = \int_{-\infty}^{\infty} \frac{\varepsilon}{(\lambda-u)^2+\varepsilon^2}\,\mathrm{d}\rho_\alpha(\lambda) \leqslant C. \tag{3}$$

If we write $U_\beta = \mathrm{Re}\, m_\beta(z)$ and $V_\beta = \mathrm{Im}\, m_\beta(z)$ and do the same for m_α

then, together with Lemma 1 we obtain

$$V_\beta(z) = \frac{[(\cot \gamma)^2 + 1]V_\alpha(z)}{[U_\alpha(z) + \cot \gamma]^2 + V_\alpha^2(z)}. \tag{4}$$

We know from the hypotheses that $\rho_\beta'(\lambda) > N^{\cdot} > 0$ for $\lambda \in (\underline{\lambda}, \bar{\lambda})$.

Exactly as in claim (b) one can prove that there exists a constant $R > 0$ such that

$$|V_\beta(z)| = |\operatorname{Im} m_\beta(u + i\varepsilon)| \geqslant R > 0$$

uniformly in ε for $u \in I'$.

From here and from (3) and (4) it follows the existence of a constant ζ such that

$$|U_\alpha(u + i\varepsilon)| \leqslant \zeta$$

uniformly for u in I'.

So far we have proved that

$$|m_\alpha(u + i\varepsilon)| < N \quad \text{and} \quad |\operatorname{Im} m_\alpha(u + i\varepsilon)| > M$$

holds for $u \in I' = [\underline{\lambda}', \bar{\lambda}']$ uniformly, for ε small. Together with Theorem 1 and the fact that the absolute continuity in each closed subinterval of $(\underline{\lambda}, \bar{\lambda}) = \mathrm{I}$ implies the absolute continuity in all I, we obtain the absolute continuity of ρ in I.

3. Applications

(I) Sturm–Liouville Operators with periodic coefficients.

As an application of Theorem 2, we consider the following problem

$$(lu)(x) = -u''(x) + q(x)u(x) \qquad 0 \leqslant x < \infty$$
$$u(0) \cos \alpha + u'(0) \sin \alpha = 0$$

with $q(x) = q(x + c)$ i.e. periodic with period c. The endpoint 0 is regular. The *l.p.c.* occurs at ∞ (see **[2]**).

Let $u_1(x, z)$ and $u_2(x, z)$ be two linearly independent solutions of $lu = zu$ which satisfy the following conditions

$$u_1(0, z) = \sin \alpha \qquad u_2(0, z) = \cos \alpha$$
$$u_1'(0, z) = -\cos \alpha \qquad u_2'(0, z) = \sin \alpha.$$

Since q is periodic, the functions $u_1(x + c, z)$ and $u_2(x + c, z)$ are also two linearly independent solutions of the equality $lu = zu$. Therefore, we can write

$$u_1(x, z) = c_{11}(z)u_1(x + c, z) + c_{12}u_2(x + c, z) \tag{5}$$
$$u_2(x, z) = c_{21}(z)u_1(x + c, z) + c_{22}u_2(x + c, z). \tag{6}$$

We can easily determine the coefficients $c_{ij}(z)$ (e.g., by differentiating (5) and (6) and evaluating at $x=0$); what we obtain is

$$\begin{cases} c_{11} = u_2(c, z)\cos\alpha + u_2'(c, z)\sin\alpha \\ c_{12} = -[u_1(c, z)\cos\alpha + u_1'(c, z)\sin\alpha] \\ c_{21} = u_2(c, z)\sin\alpha + u_2'(c, z)\cos\alpha \\ c_{22} = u_1(c, z)\sin\alpha - u_1'(c, z)\sin\alpha \end{cases} \tag{7}$$

since the *l.p.c.* occurs at ∞, there is a uniquely defined number $m(z)$ with

$$m(z)u_1(x, z) + u_2(x, z) \in L_2(0, \infty).$$

It now follows from (5) and (6) that

$$\begin{aligned} [m(z)c_{11}(z) + c_{21}(z)]u_1(x+c, z) + [m(z)c_{12}(z) + c_{22}(z)]u_2(x+c, z) \\ = m(z)u_1(x, z) + u_2(x, z) \in L_2(0, \infty). \end{aligned}$$

On the other hand, we have

$$m(z)u_1(x, z) + u_2(x, z) \in L_2$$

thus

$$m(z)u_1(x+c, z) + u_2(x+c, z) \in L_2$$

and therefore

$$\begin{aligned} m(z)[m(z)c_{12} + c_{22}]u_1(x+c, z) + [m(z)c_{12} + c_{22}]u_2(x+c, z) \\ = [m(z)c_{12} + c_{22}]\{m(z)u_1(x+c, z) + u_2(x+c, z)\} \in L_2. \end{aligned}$$

Since only one solution (up to a constant factor) of $lu = zu$ for $z \in \mathbf{C}\backslash\mathbf{R}$ can be in L_2, and since the coefficients of $u_2(x+c, z)$ in both solutions are identical, it follows that

$$m(z)c_{11}(z) + c_{21}(z) = m(z)[m(z)c_{12} + c_{22}].$$

Therefore, $m(z)$ satisfies the equality

$$c_{12}[m(z)]^2 + [c_{22} - c_{11}]m(z) - c_{21} = 0$$

i.e. $m(z)$ must be given by

$$m(z) = \frac{[c_{11}(z) - c_{22}(z)] \pm \sqrt{[c_{22}(z) - c_{11}(z)]^2 + 4c_{12}c_{21}(z)}}{2c_{12}(z)} \tag{8}$$

From the spectral theory of periodic differential operators (see [**4**] p. 27) we know that for $z_0 \in S$ we have

$$4 > (c_{11}(z_0) + c_{22}(z_0))^2,$$

where S is the set of interior stability points.

Using the expression (7) for c_{ij} and the equality $W(u_1, u_2) = 1$ we can see that $c_{11}c_{22}-c_{12}c_{21}=1$. Hence for $z\in S$

$$[c_{11}+c_{22}]^2<4[c_{11}c_{22}-c_{12}c_{21}]$$

thus,

$$[c_{22}-c_{11}]^2<-4c_{12}c_{21}$$

hence,

$$[c_{22}(z)-c_{11}(z)]^2+4c_{12}(z)c_{21}(z)<0$$

($c_{ij}(z)$ are analytic).

Let $A_\pm(z)$ be the right side of (8) (where the $\pm$ sign corresponds to the $\pm$ sign of the square root) and consider an interval $I\subset S$ such that $c_{12}(z)\neq 0$ for $z\in I$. (Since $c_{12}(z)$ is holomorphic there is only a discrete set of points where $c_{12}(z)=0$). Since $c_{ij}(z)\in\mathbf{R}$ for $z\in\mathbf{R}$ it follows that $|\operatorname{Im} A_\pm(z)|>0$ for $z\in S$. From the continuity of $\operatorname{Im} A_\pm(z)$ it follows that for each $z_0\in I$ there exists constants $M^\pm_{z_0}>0$ and circles $U^\pm_{z_0}$ with center at z_0 such that $|\operatorname{Im} A_\pm(z)|>M^\pm_{z_0}>0$ for all $z\in U^\pm_{z_0}$; also, there exist $k^\pm_{z_0}>0$ and $\delta^\pm_{z_0}>0$ such that

$$(z_0-\delta^\pm_{z_0},\, z_0+\delta^\pm_{z_0})\times(z_0,\, z_0+ik^\pm_{z_0})\subset U^\pm_{z_0}$$

holds. Therefore we have

$$|\operatorname{Im} A_\pm(u+i\varepsilon)|>M^\pm_{z_0}>0$$

for $0<\varepsilon<k$ and for $u\in(z_0-\delta^\pm,\, z_0+\delta^\pm)$.

Since I is compact we can conclude that there are constants $M>0$ and $k>0$ such that

$$|\operatorname{Im} A_\pm(u+i\varepsilon)|>M>0 \tag{9}$$

for $u\in I$ and $0<\varepsilon<k$.

Now, from the continuity of $A_\pm(z)$ in I it follows that for each $z_0\in I$ there are constants $N^\pm_{z_0}$ and circles $U^\pm_{z_0}$ with center at z_0, such that

$$|A_\pm(z)|<N^\pm_{z_0}\quad\text{for all}\quad z\in U^\pm_{z_0}$$

Analogously to what we have done, we can conclude that there are constants $N>0$, $k>0$ such that

$$|A_\pm(u+i\varepsilon)|<N\quad\text{for}\quad u\in I,\qquad 0<\varepsilon<k. \tag{10}$$

Since $m(z)=A_\pm(z)$ for $z\in\mathbf{C}\backslash\mathbf{R}$, it follows from (9) and (10) that $m(z)$ satisfies the conditions of Theorem 1 and therefore the spectral function $\rho(\lambda)$ is absolutely continuous in I.

There are no eigenvalues in S, therefore, the spectral function $\rho(\lambda)$ is continuous in S. The spectral function is moreover absolutely continuous in each interval I not containing those points z such that $c_{12}(z)=0$. Since $\rho(\lambda)$ is monotone it follows that ρ is absolutely continuous in S.

From here and from Theorem 1 it follows that S is contained in the absolutely continuous spectrum of each self-adjoint realization of the following problem ($-\infty \leqslant a \leqslant 0$):

$$\tilde{l}u = -u'' + \tilde{q}(x)u \qquad a < x < \infty$$

$\tilde{q} \in L_{1,\mathrm{loc}}(a, \infty)$, (i.e. q locally integrable)

$$\tilde{q}(x) = \begin{cases} q(x) \text{ for} & 0 \leqslant x < \infty \\ \text{arbitrary for} & a < x < 0. \end{cases}$$

As a particular case we have $q \equiv 0$ in $(0, \infty)$. Then the spectrum is absolutely continuous in $(0, \infty)$. However, this also follows from [**12**]. This result generalizes Corollary II.6 of [**1**] where the arbitrary part of $\tilde{q}$ had to be bounded from below by a function of the form $-a_0(t^2+1)$, $a_0 > 0$. The result in [**1**] was obtained with different methods.

(II) In Hinton–Shaw [**6**] the following differential expression was considered

$$\tau y = -y'' + \{q(x) + p(x)\}y \qquad -\infty < x < \infty$$

with $q(x)$ real, piecewise continuous and periodic, $p(x) \in L_1(-\infty, \infty)$. From [**6**] it can be seen (see [**3**]) that the conditions of Theorems 1 and 2 are satisfied; therefore, we immediately obtain the following result: the set of interior stability points S is contained in the absolute continuous spectrum of the problem

$$\hat{l}u = -u'' + \hat{q}(x)u - \infty < x < \infty \tag{11}$$

with $\hat{q} \in L_{1,\mathrm{loc}}(-\infty, \infty)$

$$\hat{q}(x) = \begin{cases} q(x) + p(x) & \text{for} \quad 0 \leqslant x < \infty \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

q periodic, $p \in L_1(0, \infty)$.

This generalizes our last result and also Theorem 1 of [**6**] where q had to satisfy certain conditions in $(-\infty, 0)$.

The inverse result is not true, i.e. the spectrum of the problem (11) may not be contained in S (see [**3**]).

Results similar to Theorems 1 and 2 can be proved for Dirac Systems (see [**3**]). With these analogs we can prove that in the three theorems of Hinton–Shaw [**7**] the absolutely continuous spectrum is independent of the behavior of the coefficients of the Dirac System at the left point. (In [**7**] additional conditions were needed on the behavior of the coefficients at the left point).

It was noted at the end of [**7**] (Remark 2) that there are important relations between the spectra of the operator T in $(-\infty, \infty)$ and the spectra of the self-adjoint realizations T_α^- or T_α of the Dirac System in

$(-\infty, 0]$ or $[0, \infty)$. (These operators are regular at 0). When in an interval $J \subset \mathbf{R}$, T_α^- or T_α have discrete spectrum, while the other has continuously differentiable spectrum in J with $\rho'(\lambda) > 0$, then T has continuous differentiable spectrum in J.

We remark that the assumption "T_α^- or T_α has discrete spectrum" is not necessary to obtain the absolute continuity of the spectrum of T in J.

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