ON CONDITIONS WHICH GUARANTEE
STABILITY OF THE ABSOLUTELY
CONTINUOUS SPECTRUM OF ORDINARY
DIFFERENTIAL OPERATORS

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1. Introduction

In [11] J. Weidmann proved the absolute continuity of the spectrum of
Sturm–Liouville operators under very general conditions. The following
differential expressions were considered

$$(lu)(x) = -u''(x) + q(x)u(x) \quad \text{for } x \in (a, \infty),$$

where $q$ is a real valued, locally integrable function defined in $(a, \infty)$.

In [11], aside from conditions on $q$ in the interval $(c, \infty)$ with $a < c < \infty$,
there were (implicit) conditions on the behavior of $q$ by the point $a$: to
prove the absolute continuity of the spectrum of all the self-adjoint
realizations of $l$ in an interval $(\lambda, \bar{\lambda})$ it was necessary that all self-adjoint
realizations of $l$ in $L_2(a, c)$ had only discrete spectrum in $(\lambda, \bar{\lambda})$.

The same result was proved without this additional condition by $a$ in E.
and Sturm–Liouville operators. In [5] the behavior of the resolvent near
the real axis was investigated whereas in [11] and also in [12] oscillation
methods were used.

The fact that the absolute continuity of the spectrum in these cases
cannot be perturbed by the behavior at the left point $a$, suggests the
following conjecture (which goes back essentially to J. Weidmann [12]):
Let $l$ be a formally self-adjoint differential expression in $(a, b)$ and $A$ a
self-adjoint realization. For $c \in (a, b)$ let each self-adjoint realization $A_b$
of $l$ in $(c, b)$ have absolutely continuous spectrum in $(\lambda, \bar{\lambda})$. Then $A$ also
has absolutely continuous spectrum in $(\lambda, \bar{\lambda})$. (We say that a self-adjoint
operator has absolutely continuous spectrum in an interval $I$ when
$\|E(\lambda)f\|^2$ is absolutely continuous in $I$ with respect to the Lebesgue
measure for all $f \in L_2(a, b)$, $E(\lambda)$ being the family of spectral projections
of the operator).

It was proved in [3], using the theory of Weyl–Titchmarsh–Kodaira
and the theorem of Gelfand–Levitan, that the absolute continuity of all
the self-adjoint realizations $A_b$ is not enough to imply the absolute
continuity of $A$ and so the above conjecture was refuted.

In this paper we shall prove that it is enough to give conditions on two

self-adjoint realizations of \( l \) in \((c, b)\) in order to have absolute continuity for every self-adjoint realization of \( l \) in \((a, b)\). We then apply this result to Sturm–Liouville expressions with coefficients periodic in \((c, \infty)\) and arbitrary (only locally integrable) in \((a, c)\). Corollary II.6 of Carmona [1] is a particular case of this application. We shall also give a generalization of a result of Hinton–Shaw for perturbed periodic potentials.

The above results can be extended to Dirac Systems. As an application we prove that the absolutely continuous spectrum of the Dirac Systems investigated by Hinton–Shaw [7] is independent of the behavior of the coefficients by the left point \(a\).

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2. Main result

Let \( l \) be the following differential expression

\[
lu = -u'' + q(x)u \quad a < x < b \quad 0 \in (a, b)
\]  

(1)

(it is possible to have \(a = -\infty\) or \(b = \infty\)) where \(q\) is a real-valued, locally integrable function defined in \((a, b)\). We will also assume that the limit point case \((l.p.c.)\) occurs at \(b\).

Define the operators \(L_\theta\) and \(L\) as follows:

\[
D(L_\theta) = \{u \in L_2(0, b) \mid u, u' \text{ absolutely continuous} \in (0, b), \, lu \in L_2(0, b) \, u(0) \cos \theta + u'(0) \sin \theta = 0\}
\]

(since the endpoint 0 is regular we have the limit circle case \((l.c.c.)\) there)

\[L_\theta u = lu\]

\[
D(L) = \{u \in L_2(a, b) \mid u, u' \text{ absolutely continuous} \in (a, b), \, lu \in L_2(a, b) \, [u, v]_a = 0 \text{ when } l.c.c. \text{ at } a\}
\]

\((v)\) is a nontrivial real solution of \(lu = 0\)

\[Lu = lu.
\]

The operators \(L_\theta\) and \(L\) are self-adjoint in \(L_2(0, b)\) and \(L_2(a, b)\), respectively.

We know (see [14] p. 257) that for \(z \in \mathbf{C} \setminus \mathbf{R}\) the Resolvent of \(L\) has the form

\[
R_z(L)g(x) = (z - L)^{-1}g(x)
\]

\[= W(u_a, u_b)^{-1}\left\{u_b(x) \int_a^x u_a(y)g(y) \, dy + u_a(x) \int_x^b u_b(y)g(y) \, dy\right\}
\]
where $W(\cdot, \cdot)$ is the Wronskian and $u_a$, $u_b$ are solutions of $lu = zu$, uniquely determined up to a constant factor by the conditions: $u_a$ is in $L_2(a, c)$ if we have l.p.c. at $a$ or $u_a$ satisfies the boundary condition at $a$ if l.c.c. takes place at $a$; the function $u_b$ is in $L_2(c, b)$.

Let $u_1, u_2$ be a fundamental set of solutions of $lu = zu$ satisfying

$$
\begin{align*}
  u_1(0, z) &= \sin \theta \\
  u_1'(0, z) &= -\cos \theta \\
  u_2(0, z) &= \cos \theta \\
  u_2'(0, z) &= \sin \theta \\
  \theta &\in [0, 2\pi).
\end{align*}
$$

For $z \in \mathbb{C} \setminus \mathbb{R}$ $W(u_1, u_a)(z) \neq 0$ and $W(u_1, u_b)(z) \neq 0$ since otherwise we would have a self-adjoint problem with a nonreal eigenvalue.

Choose $m_b(z)$ and $m_a(z)$ such that

$$
u_b(x, z) = m_b(z)u_1(x, z) + u_2(x, z)$$

and $m_a(z)$ satisfying an analogous equality. Then, we can write

\[
R_z(L)y(x) = [W(m_a(z)u_1(x, z) + u_2(x, z)), (m_b(z)u_1(x, z) + u_2(x, z))]^{-1}
\]

\[
\cdot \left\{(m_b(z)u_1(x, z) + u_2(x, z)) \int_x^a (m_a(z)u_1(y, z)
+ u_2(y, z))g(y) \, dy + (m_a(z)u_1(x, z)
+ u_2(x, z)) \int_x^b (m_b(z)u_1(y, z) + u_2(y, z))g(y) \, dy\right\}
\]

This integral operator has the kernel

\[
R(x, y, z) = \begin{cases} 
\sum_{j, k=1}^2 m_{jk}^+(z)u_j(x, z)u_k(y, z) & \text{for } y < x \\
\sum_{j, k=1}^2 m_{jk}^-(z)u_j(x, z)u_k(y, z) & \text{for } x < y
\end{cases}
\]

where

\[
\begin{align*}
(m_{11}^+(z) & \quad m_{12}^+(z)) = \begin{pmatrix}
m_a(z) m_b(z) & m_b(z) \\
W(u_a, u_b)(z) & W(u_a, u_b)(z)
\end{pmatrix} \\
(m_{21}^+(z) & \quad m_{22}^+(z)) = \begin{pmatrix}
m_a(z) m_b(z) & m_a(z) \\
W(u_a, u_b)(z) & W(u_a, u_b)(z)
\end{pmatrix} \\
(m_{11}^-) & \quad m_{12}^-) = \begin{pmatrix}
m_a(z) m_b(z) & m_a(z) \\
W(u_a, u_b)(z) & W(u_a, u_b)(z)
\end{pmatrix} \\
(m_{21}^-) & \quad m_{22}^-) = \begin{pmatrix}
m_b(z) & m_b(z) \\
W(u_a, u_b)(z) & W(u_a, u_b)(z)
\end{pmatrix}
\]

\[
W(u_a, u_b) = [m_a - m_b]W(u_1, u_2)
\]
We call the matrices \((m_{ij}^\pm)\) the characteristic matrices of the operator \(L\) corresponding to the system \(\{u_1, u_2\}\). If we choose \(u_1 = u_a\) we obtain the following characteristic matrices for \(L_{\theta}\):

\[
\begin{pmatrix}
m_{11}^+ & m_{12}^+ \\
m_{21}^+ & m_{22}^+
\end{pmatrix} = \begin{pmatrix}
m_b & 0 \\
W(u_1, u_2) & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
W(u_1, u_2)
\end{pmatrix}
\]

\[
\begin{pmatrix}
m_{11}^- & m_{12}^- \\
m_{21}^- & m_{22}^-
\end{pmatrix} = \begin{pmatrix}
m_b & 1 \\
W(u_1, u_2) & W(u_1, u_2)
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

Let \(\rho_{\theta}\) and \(\rho\) be the \((2 \times 2\) matrix valued\) spectral functions of \(L_{\theta}\) and \(L\).

The operator \(L\) is unitarily equivalent to the operator of multiplication by the variable in \(L_2(\mathbb{R}, d\rho)\). The spectral function is given by the formula of Weyl–Titchmarsh–Kodaira

\[
\rho_{kj}(\lambda) = -\frac{1}{\pi} \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} \int_{\delta}^{\lambda + \delta} \text{Im} \, m_{kj}^\pm(s + i\epsilon) \, ds.
\]

The same is true for \(L_{\theta}\) and its spectral function.

For the fundamental system \(\{u_1, u_2\}\), with \(u_1 = u_a\) we have that the spectral function \(\rho_{\theta} = (\rho_{\theta})_g\) has only one component different from zero (abusing of the notation we denote this component by \(\rho_{\theta}\)) and

\[
-\text{Im} \, m_b(z) = \int_{-\infty}^{\infty} \frac{\epsilon}{(u - \lambda)^2 + \epsilon^2} d\rho_{\theta}(\lambda) \quad z = u + i\epsilon.
\]

(See \([8]\) p. 136 Theorem 5.2).

**Theorem 1.** If there exists constants \(k > 0\), \(N > 0\), \(M > 0\) such that the inequalities:

\[
|m_b(u + i\epsilon)| < N
\]

and

\[
|\text{Im} \, m_b(u + i\epsilon)| > M
\]

hold for \(0 < \epsilon < k\) and for all \(u \in I = [\bar{\lambda}, \bar{\lambda}]\), then \(\rho(\lambda)\) is absolutely continuous in \(I\).

**Proof.** Let

\[
a(z) = \text{Re} \, m_a(z) \quad c(z) = \text{Re} \, m_b(z)
\]

\[
b(z) = \text{Im} \, m_a(z) \quad d(z) = \text{Im} \, m_b(z).
\]

\[
-\text{Im} \, m_{22}^+ = -\text{Im} \, \frac{1}{m_a(z) - m_b(z)} = \frac{b - d}{(a - c)^2 + (b - d)^2}.
\]
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It is known that \( b(z) > 0 \) and \( d(z) < 0 \) when \( \text{Im } z > 0 \). (See [10]). Therefore, we have

\[
0 < \frac{b - d}{(a - c)^2 + (b - d)^2} \leq -\frac{1}{\text{d}} = -\frac{1}{\text{Im } m_b(z)},
\]

and hence

\[
-\text{Im} \frac{1}{m_a(u + i\varepsilon) - m_b(u + i\varepsilon)} \leq \frac{1}{|\text{Im } m_b(u + i\varepsilon)|}.
\]

Now, we know that for \( 0 < \varepsilon < k, |\text{Im } m_a(u + i\varepsilon)| > M \) holds; hence,

\[
-\text{Im} \frac{1}{m_a(u + i\varepsilon) - m_b(u + i\varepsilon)} < \frac{1}{M}
\]

for \( u \in I \).

Therefore, we have

\[
\rho_{22}(\mu) - \rho_{22}(\gamma) = \frac{1}{\pi} \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{\gamma + \delta}^{\mu + \delta} \left\{ -\text{Im} \frac{1}{m_a(u + i\varepsilon) - m_b(u + i\varepsilon)} \right\} du
\]

\[
\leq \frac{1}{\pi M} \int_{\gamma}^{\mu} du,
\]

i.e. \( \rho_{22} \) is absolutely continuous.

Now consider

\[
-\text{Im} m_{11}^+(z) = -\text{Im} \frac{m_am_b}{m_a - m_b}
\]

\[
= \frac{\text{Im } \frac{1}{m_b} - \text{Im } \frac{1}{m_a}}{\left( \text{Re } \frac{1}{m_b} - \text{Re } \frac{1}{m_a} \right)^2 + \left( \text{Im } \frac{1}{m_b} - \text{Im } \frac{1}{m_a} \right)^2}
\]

\((>0 \text{ for } z \text{ in } \mathbb{C} \text{ with } \text{Im } z > 0)\)

\[
\leq \frac{1}{\text{Im } \frac{1}{m_b} - \text{Im } \frac{1}{m_a}} \leq -\frac{|m_b|}{\text{Im } m_b} \leq \frac{N}{M}.
\]

Hence,

\[
\rho_{11}(\mu) - \rho_{11}(\gamma) \leq \int_{\gamma}^{\mu} du \cdot \frac{1}{\frac{N}{\pi M}}
\]

and therefore \( \rho_{11} \) is absolutely continuous.
One may also see [8, page 155] that
\[
\left( \text{Im} \frac{m_a}{m_a - m_b} \right)^2 = \left( \text{Im} \frac{m_b}{m_a - m_b} \right)^2 \leq \left( \text{Im} \frac{1}{m_a - m_b} \right) \left( \text{Im} \frac{m_a m_b}{m_a - m_b} \right)
\]
holds. From here the absolute continuity of \( \rho_{12} \) and \( \rho_{21} \) follows.

The function \( m_b \) depends on \( \{u_1, u_2\} \) which depend on \( \theta \). Let us now forget about the dependence on \( b \) and write \( m_\theta \) instead of \( m_b \). Then we have the following.

**Lemma 1.** Let \( \alpha \) and \( \beta \) be two different values of \( \theta \), i.e. \( \alpha \in [0, 2\pi) \), \( \beta \in [0, 2\pi) \) and \( \alpha \neq \beta \). Then we have following equality:
\[
m_\beta(z) = \frac{m_\alpha(z) \cot \gamma - 1}{m_\alpha(z) + \cot \gamma}, \quad \gamma = \alpha - \beta
\]
(See [10]).

Now we can give conditions on \( \rho_\theta \) which guarantee the absolute continuity of \( \rho \).

**Theorem 2.** Let \( \rho_\theta \) be absolutely continuous in \( I = (\lambda, \bar{\lambda}) \subset R \) for \( \theta = \alpha \) and \( \theta = \beta \) (two different values of \( \theta \)). Let \( M, N, N' \) be constants such that the inequalities
\[
0 < N < \frac{d\rho_\alpha}{d\lambda} \bigg|_{\lambda \in I} < M < \infty
\]
and
\[
0 < N' < \frac{d\rho_\beta}{d\lambda} \bigg|_{\lambda \in I}
\]
hold.

Then \( \rho \) is absolutely continuous in \( I \).

**Proof.** First we prove two claims:

Claim (a). There exists \( k > 0 \) such that for \( 0 < \varepsilon < k \) we have
\[
\int_\lambda^{\bar{\lambda}} \frac{\varepsilon}{(\lambda - u)^2 + \varepsilon^2} d\rho_\alpha(\lambda) < C = \text{constant}
\]
uniformly for \( u \in I' = [\lambda', \bar{\lambda}'] \subset I = (\lambda, \bar{\lambda}) \).

**Proof.** First, we consider the integral over \( I = [\lambda, \bar{\lambda}] \)
\[
\int_\lambda^{\bar{\lambda}} \frac{\varepsilon}{(\lambda - u)^2 + \varepsilon^2} d\rho_\alpha(\lambda) = \int_\lambda^{\bar{\lambda}} \frac{\varepsilon}{(\lambda - u)^2 + \varepsilon^2} \rho'_\alpha(\lambda) d\lambda
\]
\[
\leq M \int_\lambda^{\bar{\lambda}} \frac{\varepsilon}{(\lambda - u)^2 + \varepsilon^2} d\lambda \leq M \pi.
\]
Now let \( u \in I' \). Then we have

\[
\int_{I'} \frac{\varepsilon}{(\lambda - u)^2 + \varepsilon^2} \, d\rho_\alpha(\lambda) \leq \varepsilon C_1 \int_{I'} \frac{1}{1 + \lambda^2} \, d\rho_\alpha(\lambda) \leq \varepsilon C_1 C_2
\]

because there exists a constant \( C_1 > 0 \) such that

\[
\frac{1}{(\lambda - u)^2} \leq \frac{C_1}{1 + \lambda^2}
\]

for every \( u \in I' \), \( \lambda \in \mathbb{R} \setminus I' \) and

\[
\int_{-\infty}^{\infty} \frac{1}{1 + \lambda^2} \, d\rho_\alpha(\lambda) = -\text{Im} \, m_\alpha(i) = C_2 < \infty
\]

(see formula (2)). This proves our claim.

**Claim (b).** There exists \( k \) such that for \( 0 < \varepsilon < k \) we have

\[
\int_{I'} \frac{\varepsilon}{(\lambda - u)^2 + \varepsilon^2} \, d\rho_\alpha(\lambda) \geq D = \text{constant} > 0
\]

uniformly for \( u \in I' \).

**Proof.** We consider first the integral over \( I' \)

\[
\int_{I'} \frac{\varepsilon}{(\lambda - u)^2 + \varepsilon^2} \, d\rho_\alpha(\lambda) = \int_{I'} \frac{\varepsilon}{(\lambda - u)^2 + \varepsilon^2} \rho'_\alpha(\lambda) \, d\lambda
\]

\[
\geq N \int_{(\lambda - u)/\varepsilon} \frac{d\lambda}{v^2 + 1} \geq NC = D.
\]

(\( C > 0 \) when \( \varepsilon < k \)).

Above we have made the change of variables \( v = (\lambda - u)/\varepsilon \). Also, since \( \rho_\alpha \) does not decrease we have

\[
\int_{I'} \frac{\varepsilon}{(\lambda - u)^2 + \varepsilon^2} \, d\rho_\alpha(\lambda) \geq 0.
\]

From this inequality claim (b) follows.

Using formula (2) we can write

\[
D \leq | -\text{Im} \, m_\alpha(u + i\varepsilon)| = \int_{-\infty}^{\infty} \frac{\varepsilon}{(\lambda - u)^2 + \varepsilon^2} \, d\rho_\alpha(\lambda) \leq C.
\]

If we write \( U_\beta = \text{Re} \, m_\beta(z) \) and \( V_\beta = \text{Im} \, m_\beta(z) \) and do the same for \( m_\alpha \).
then, together with Lemma 1 we obtain

\[ V_\beta(z) = \frac{[(\cot \gamma)^2 + 1]V_\alpha(z)}{[U_\alpha(z) + \cot \gamma]^2 + V_\alpha^2(z)}. \]  

(4)

We know from the hypotheses that \( \rho_\beta'(\lambda) > N' > 0 \) for \( \lambda \in (\lambda, \tilde{\lambda}) \).

Exactly as in claim (b) one can prove that there exists a constant \( R > 0 \) such that

\[ |V_\beta(z)| = |\text{Im} \ m_\beta(u + i\varepsilon)| \geq R > 0 \]

uniformly in \( \varepsilon \) for \( u \in I' \).

From here and from (3) and (4) it follows the existence of a constant \( \zeta \) such that

\[ |U_\alpha(u + i\varepsilon)| \leq \zeta \]

uniformly for \( u \in I' \).

So far we have proved that

\[ |m_\alpha(u + i\varepsilon)| < N \quad \text{and} \quad |\text{Im} m_\alpha(u + i\varepsilon)| > M \]

holds for \( u \in I' = [\lambda', \tilde{\lambda}'] \) uniformly, for \( \varepsilon \) small. Together with Theorem 1 and the fact that the absolute continuity in each closed subinterval of \( (\lambda, \tilde{\lambda}) = I \) implies the absolute continuity in all \( I \), we obtain the absolute continuity of \( \rho \) in \( I \).

3. Applications

(I) Sturm–Liouville Operators with periodic coefficients.

As an application of Theorem 2, we consider the following problem

\[ (lu)(x) = -u''(x) + q(x)u(x) \quad 0 \leq x < \infty \]

\[ u(0) \cos \alpha + u'(0) \sin \alpha = 0 \]

with \( q(x) = q(x + c) \) i.e. periodic with period \( c \). The endpoint 0 is regular. The l.p.c. occurs at \( \infty \) (see [2]).

Let \( u_1(x, z) \) and \( u_2(x, z) \) be two linearly independent solutions of \( lu = zu \) which satisfy the following conditions

\[ u_1(0, z) = \sin \alpha \quad u_2(0, z) = \cos \alpha \]
\[ u_1'(0, z) = -\cos \alpha \quad u_2'(0, z) = \sin \alpha. \]

Since \( q \) is periodic, the functions \( u_1(x + c, z) \) and \( u_2(x + c, z) \) are also two linearly independent solutions of the equality \( lu = zu \). Therefore, we can write

\[ u_1(x, z) = c_{11}(z)u_1(x + c, z) + c_{12}u_2(x + c, z) \]  

(5)

\[ u_2(x, z) = c_{21}(z)u_1(x + c, z) + c_{22}u_2(x + c, z). \]  

(6)
We can easily determine the coefficients $c_{ij}(z)$ (e.g., by differentiating (5) and (6) and evaluating at $x = 0$); what we obtain is

\[
\begin{align*}
    c_{11} &= u_2(c, z) \cos \alpha + u'_2(c, z) \sin \alpha \\
    c_{12} &= -[u_1(c, z) \cos \alpha + u'_1(c, z) \sin \alpha] \\
    c_{21} &= u_2(c, z) \sin \alpha + u'_2(c, z) \cos \alpha \\
    c_{22} &= u_1(c, z) \sin \alpha - u'_1(c, z) \sin \alpha
\end{align*}
\] (7)

since the l.p.c. occurs at $\infty$, there is a uniquely defined number $m(z)$ with

\[m(z)u_1(x, z) + u_2(x, z) \in L_2(0, \infty).\]

It now follows from (5) and (6) that

\[
[m(z)c_{11}(z) + c_{21}(z)]u_1(x + c, z) + [m(z)c_{12}(z) + c_{22}(z)]u_2(x + c, z)
= m(z)[m(z)c_{11}(z) + c_{21}(z)]u_1(x, z) + u_2(x, z) \in L_2(0, \infty).
\]

On the other hand, we have

\[m(z)u_1(x, z) + u_2(x, z) \in L_2\]

thus

\[m(z)u_1(x + c, z) + u_2(x + c, z) \in L_2\]

and therefore

\[
m(z)[m(z)c_{12} + c_{22}]u_1(x + c, z) + [m(z)c_{12} + c_{22}]u_2(x + c, z)
= [m(z)c_{12} + c_{22}][m(z)u_1(x + c, z) + u_2(x + c, z)] \in L_2.
\]

Since only one solution (up to a constant factor) of $lu = zu$ for $z \in \mathbb{C}\setminus\mathbb{R}$ can be in $L_2$, and since the coefficients of $u_2(x + c, z)$ in both solutions are identical, it follows that

\[m(z)c_{11}(z) + c_{21}(z) = m(z)[m(z)c_{12} + c_{22}].\]

Therefore, $m(z)$ satisfies the equality

\[c_{12}[m(z)]^2 + [c_{22} - c_{11}]m(z) - c_{21} = 0\]

i.e. $m(z)$ must be given by

\[m(z) = \frac{[c_{11}(z) - c_{22}(z)] \pm \sqrt{[c_{22}(z) - c_{11}(z)]^2 + 4c_{12}c_{21}(z)}}{2c_{12}(z)}\] (8)

From the spectral theory of periodic differential operators (see [4] p. 27) we know that for $z_0 \in S$ we have

\[4 > (c_{11}(z_0) + c_{22}(z_0))^2,
\]

where $S$ is the set of interior stability points.

Using the expression (7) for $c_{ij}$ and the equality $W(u_1, u_2) = 1$ we can
see that \(c_{11}c_{22} - c_{12}c_{21} = 1\). Hence for \(z \in S\)

\[
[c_{11} + c_{22}]^2 < 4[c_{11}c_{22} - c_{12}c_{21}]
\]

thus,

\[
[c_{22} - c_{11}]^2 < -4c_{12}c_{21}
\]

hence,

\[
[c_{22}(z) - c_{11}(z)]^2 + 4c_{12}(z)c_{21}(z) < 0
\]

\((c_{ij}(z)\) are analytic).

Let \(A_{\pm}(z)\) be the right side of (8) (where the \(\pm\) sign corresponds to the
\(\pm\) sign of the square root) and consider an interval \(I \subset S\) such that
\(c_{12}(z) \neq 0\) for \(z \in I\). (Since \(c_{12}(z)\) is holomorphic there is only a discrete
set of points where \(c_{12}(z) = 0\)). Since \(c_{ij}(z) \in \mathbb{R}\) for \(z \in \mathbb{R}\) it follows that
\(|\text{Im} A_{\pm}(z)| > 0\) for \(z \in S\). From the continuity of \(\text{Im} A_{\pm}(z)\) it follows that
for each \(z_0 \in I\) there exists constants \(M_{z_0}^\pm > 0\) and circles \(U_{z_0}^\pm\) with center at
\(z_0\) such that \(|\text{Im} A_{\pm}(z)| > M_{z_0}^\pm > 0\) for all \(z \in U_{z_0}^\pm\); also, there exist \(k_{z_0}^\pm > 0\)
and \(\delta_{z_0}^\pm > 0\) such that

\[
(z_0 - \delta_{z_0}^\pm, z_0 + \delta_{z_0}^\pm) \times (z_0, z_0 + ik_{z_0}) \subset U_{z_0}^\pm
\]

holds. Therefore we have

\[
|\text{Im} A_{\pm}(u + i\varepsilon)| > M_{z_0}^\pm > 0
\]

for \(0 < \varepsilon < k\) and for \(u \in (z_0 - \delta^\pm, z_0 + \delta^\pm)\).

Since \(I\) is compact we can conclude that there are constants \(M > 0\) and
\(k > 0\) such that

\[
|\text{Im} A_{\pm}(u + i\varepsilon)| > M > 0
\]  \(\text{(9)}\)

for \(u \in I\) and \(0 < \varepsilon < k\).

Now, from the continuity of \(A_{\pm}(z)\) in \(I\) it follows that for each \(z_0 \in I\)
there are constants \(N_{z_0}^\pm\) and circles \(U_{z_0}^\pm\) with center at \(z_0\), such that

\[
|A_{\pm}(z)| < N_{z_0}^\pm \quad \text{for all} \quad z \in U_{z_0}^\pm
\]

Analogously to what we have done, we can conclude that there are
constants \(N > 0\), \(k > 0\) such that

\[
|A_{\pm}(u + i\varepsilon)| < N \quad \text{for} \quad u \in I, \quad 0 < \varepsilon < k. \quad \text{(10)}
\]

Since \(m(z) = A_{\pm}(z)\) for \(z \in \mathbb{C} \setminus \mathbb{R}\), it follows from (9) and (10) that \(m(z)\)
satisfies the conditions of Theorem 1 and therefore the spectral function
\(\rho(\lambda)\) is absolutely continuous in \(I\).

There are no eigenvalues in \(S\), therefore, the spectral function \(\rho(\lambda)\) is
continuous in \(S\). The spectral function is moreover absolutely continuous
in each interval \(I\) not containing those points \(z\) such that \(c_{12}(z) = 0\). Since
\(\rho(\lambda)\) is monotone it follows that \(\rho\) is absolutely continuous in \(S\).

From here and from Theorem 1 it follows that \(S\) is contained in the
absolutely continuous spectrum of each self-adjoint realization of the following problem \((-\infty \leq a \leq 0)\):

\[
\hat{u} = -u'' + \hat{q}(x)u \quad a < x < \infty
\]

\[
\hat{q} \in L_{1,\text{loc}}(a, \infty), \text{ (i.e. } q \text{ locally integrable)}
\]

\[
\hat{q}(x) = \begin{cases} 
q(x) & \text{for } 0 \leq x < \infty \\
\text{arbitrary} & \text{for } a < x < 0.
\end{cases}
\]

As a particular case we have \(q \equiv 0\) in \((0, \infty)\). Then the spectrum is absolutely continuous in \((0, \infty)\). However, this also follows from [12]. This result generalizes Corollary II.6 of [1] where the arbitrary part of \(\hat{q}\) had to be bounded from below by a function of the form \(-a_0(i^2 + 1), a_0 > 0\). The result in [1] was obtained with different methods.

(II) In Hinton–Shaw [6] the following differential expression was considered

\[
\tau y = -y'' + (q(x) + p(x))y \quad -\infty < x < \infty
\]

with \(q(x)\) real, piecewise continuous and periodic, \(p(x) \in L_1(-\infty, \infty)\). From [6] it can be seen (see [3]) that the conditions of Theorems 1 and 2 are satisfied; therefore, we immediately obtain the following result: the set of interior stability points \(S\) is contained in the absolute continuous spectrum of the problem

\[
\hat{u} = -u'' + \hat{q}(x)u - \infty < x < \infty
\]

\[
\hat{q} \in L_{1,\text{loc}}(-\infty, \infty)
\]

\[
\hat{q}(x) = \begin{cases} 
q(x) + p(x) & \text{for } 0 \leq x < \infty \\
\text{arbitrary} & \text{otherwise}
\end{cases}
\]

\(q\) periodic, \(p \in L_1(0, \infty)\).

This generalizes our last result and also Theorem 1 of [6] where \(q\) had to satisfy certain conditions in \((-\infty, 0)\).

The inverse result is not true, i.e.
the spectrum of the problem (11) may not be contained in \(S\) (see [3]).

Results similar to Theorems 1 and 2 can be proved for Dirac Systems (see [3]). With these analogs we can prove that in the three theorems of Hinton–Shaw [7] the absolutely continuous spectrum is independent of the behavior of the coefficients of the Dirac System at the left point. (In [7] additional conditions were needed on the behavior of the coefficients at the left point).

It was noted at the end of [7] (Remark 2) that there are important relations between the spectra of the operator \(T\) in \((-\infty, \infty)\) and the spectra of the self-adjoint realizations \(T^-\alpha\) or \(T^\alpha\) of the Dirac System in
\((-\infty, 0]\) or \([0, \infty)\). (These operators are regular at 0). When in an interval \(J \subset \mathbb{R}\), \(T_\alpha^\pm\) or \(T_\alpha\) have discrete spectrum, while the other has continuously differentiable spectrum in \(J\) with \(\rho'(\lambda) > 0\), then \(T\) has continuous differentiable spectrum in \(J\).

We remark that the assumption "\(T_\alpha^\pm\) or \(T_\alpha\) has discrete spectrum" is not necessary to obtain the absolute continuity of the spectrum of \(T\) in \(J\).

REFERENCES