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On a Problem of P. Hartman and A. Wintner

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Abstract. The relation between the set where a Sturm–Liouville problem does not have square integrable solutions and the essential spectrum is studied. A characterization of the spectrum is given when the spectrum is a perfect set.

1 Introduction

Some time ago I was trying to understand the behavior of the spectrum of Sturm–Liouville operators under local perturbations to the potential. The interesting spectrum was the singular spectrum since the absolutely continuous part is well known to be stable under this kind of perturbations. The singular continuous part resulted to be extremely unstable and it was the turn of the pure point part. Was dense point spectrum stable?. This question was studied in several papers [1] [2]. Before a result about unstability in this case was obtained, I came accross a strange set which had the property of not accepting eigenvalues for any local perturbation [3]. Later on, thanks to a conversation with Prof. H. Kalf, I realized this result could throw some light on an old problem of P. Hartman and A. Wintner [4].

The problem is to see whether two sets are always equal. Using the result about the forbidden set for eigenvalues mentioned above, one can prove that when the spectrum is a perfect set, that is, when the problem has only essential spectrum, the two sets of P. Hartman and A. Wintner coincide. That these sets can not always be identical follows easily from a theorem on inverse spectral theory due to Gelfand and Levitan (see [5, p. 282]). This will be made precise in what follows.

2 The Problem

Consider the differential expression

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$$lu = -u'' + q(x)u \qquad 0 \leq x < \infty$$

where q is a real valued, locally integrable function.

By definition, a solution of

$$lu = \lambda u$$

is a continuously differentiable function $u : [0, \infty) \to \mathbb{R}$ such that u' is absolutely continuous and the equation holds almost everywhere. λ is a real parameter.

The differential expression l generates selfadjoint operators L_α as follows:

$$L_\alpha u = lu$$

$$\begin{aligned} D(L_\alpha) \quad = \quad & \{u \in L_2(0,\infty) | u, u' \text{ are locally absolutely continuous,} \\ & lu \in L_2(0,\infty) \text{ and } u(0)\cos\alpha + u'(0)\sin\alpha = 0\}, \quad \alpha \in [0,\pi) \end{aligned}$$

The spectrum of L_α is the set

$$\begin{aligned} \sigma = \sigma(L_\alpha) : \quad = \quad & \{\lambda \in \mathbb{R} | (\lambda I - L_\alpha)^{-1} \\ & \text{does not exist or if it exists is not bounded}\} \end{aligned}$$

The essential spectrum, denoted $\sigma_{ess}(L_\alpha)$ is the set of limit points of $\sigma(L_\alpha)$.

It is well known [6, p. 38] that the essential spectrum does not depend on α, therefore we shall write $\sigma_{ess}(L_\alpha) = \sigma_{ess}$

Let us denote by S^* the complement of σ_{ess}, that is

$$S^* := \sigma_{ess}^c = \mathbb{R} \backslash \sigma_{ess}.$$

and by S_0 the interior of the set

$$\begin{aligned} S = \{\lambda \in \mathbb{R} \quad | \quad & \exists u \text{ solution of } lu = \lambda u \\ & \text{such that } \int_0^\infty u^2(t)dt < \infty\} \end{aligned}$$

In 1949 P. Hartman and A. Wintner [4] attempted to solve the following:

Problem

Is always $S_0 = S^*$?

3 The Solution

In M.S.P. Eastham & H. Kalf [6, p. 42] it is remarked that

$$♣ \qquad S_0 = S^* \Longleftrightarrow S \cap \sigma_{ess} \text{ has empty interior}$$

Part (b) of the following result contradicts one direction of ♣.

Theorem

a) $S_0 = S^* \Longrightarrow S \cap \sigma_{ess}$ has empty interior.

b) $S \cap \sigma_{ess}$ has empty interior $\not\Longrightarrow S_0 = S^*$.

c) It is not always true $S_0 = S^*$.

d) If σ is a perfect set, then $S_0 = S^*$.

Proof

a) If $S \cap \sigma_{ess}$ does not have empty interior there exists an open interval $I \subset S \cap \sigma_{ess}$. Therefore $I \subset S_0 \cap \sigma_{ess}$ and $S_0 \neq S^*$.

b) Consider the case where the essential spectrum has an isolated point p. This point p has to be limit point of isolated points of the spectrum which have to be eigenvalues. If $p \in S$ then $p \in S_0$, since points in the resolvent set are in S (see [6] p. 39). Therefore $p \in S_0 \cap \sigma_{ess}$ and $S_0 \neq S^*$.

We know, that it is possible to construct an operator L_α which in a given interval has an isolated point of σ_{ess}, as required above. This follows from the inverse theorem of Gelfand and Levitan see [5]. In fact this teorem allow us to consider any spectral situation locally. See Remark 2.

c) It is immediate from the proof of b).

d) We have only to prove $S_0 \subset S^*$ since it is always true $S_0 \supset S^*$ (see [6], p. 39). Assume this is not the case, that is $S_0 \cap \sigma_{ess} \neq \phi$, then there exists an interval $I \subset S_0$ such that $I \cap \sigma_{ess} \neq \phi$.

If we consider a closed interval J contained in I such that $J \cap \sigma_{ess} = J \cap \sigma \neq \phi$ we may apply the theorem in [3] (see Remark 2) and conclude that $J \cap S^c \neq \phi$. This is a contradiction since $J \subset S_0 \subset S$.

QED

Remarks

1. From d) we have the following characterization of the essential spectrum when the spectrum is a perfect set, i.e. when $\sigma = \sigma_{ess}$.

$$\sigma = \sigma_{ess} = S_0^c = \overline{\{\lambda \in \mathbb{R} / lu = \lambda u \text{ does not have solutions in} L^2(0,\infty)\}}$$

The line above a set denotes closure. It is clear that the set on the right does not depend on the boundary condition α.

2. It is possible to have points of σ_{ess} in S which are not isolated points of σ_{ess}. To see this consider the triadic Cantor set C. The set C is the subset of [0, 1] consisting of all numbers of the form $\sum_{n=0}^{\infty} c_n/3^n$, with $c_n = 0$ or 2; its complement in [0, 1] is a countable union of non overlapping intervals. Let $\{S_n\}_{n=1}^{\infty}$ be the set which consists of the middle points of these intervals.

Choose S_1 to be the middle point of the interval $(1/3, 2/3)$, S_2 and S_3 the middle points of the intervals $(1/3^2, 2/3^2)$ and $(7/3^2, 8/3^2)$ respectively and so on.

Let us consider a spectral function ρ which satisfies

$$\int_{\mathbb{R}\setminus I} \frac{d\rho(\lambda)}{(x-\lambda)^2} < \infty$$

where $I = [0,1]$ and define ρ to be a discrete measure in I as follows:

$$\rho(\{S_n\}) = \frac{1}{4 \cdot 3^{2n} n^2} .$$

Then

$$\int_I \frac{d\rho(\lambda)}{(x-\lambda)^2} = \sum_{n=1}^{\infty} \frac{\rho(\{S_n\})}{(x-S_n)^2}$$

$$\leq \sum_{n=1}^{\infty} 4 \cdot 3^{2n} \rho(\{S_n\}) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

if $x \in C$, the Cantor set, since in this case $|x - S_n| \geq \frac{1}{2 \cdot 3^n}$.

Moreover

$$\begin{aligned}\rho(I) &= \sum_{n=1}^{\infty} \rho(\{S_n\}) = \sum \frac{1}{4 \cdot 3^{2n} n^2} \\ &\leq \sum \frac{1}{n^2} < \infty.\end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{(x-\lambda)^2} < \infty \quad \text{if } x \in C.$$

According to a theorem of Aronszajn [7] $C \subset S$ and it follows that $[0,1] \subset S$ and $(0,1) \subset S_0$. Therefore it is not possible to have $S_0 = \sigma_{ess}^c$.

The theorem in [3] states the following: If for an interval J the set $A := J \cap \sigma$ is perfect, then $A \cap S^c$ is an uncountable set.

The example above shows that σ can not be replaced by σ_{ess} in the statement of the theorem.

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