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Quantum Dynamical Applications of Salem's Theorem

DAVID DAMANIK[1] and RAFAEL DEL RIO[2]
[1]*Department of Mathematics, Rice University, Houston, TX 77005, USA. e-mail: damanik@rice.edu*
[2]*IIMAS, UNAM, 04510 Mexico DF, Mexico. e-mail: delrio@leibniz.iimas.unam.mx*

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Abstract. We consider the survival probability of a state that evolves according to the Schrödinger dynamics generated by a self-adjoint operator H. We deduce from a classical result of Salem that upper bounds for the Hausdorff dimension of a set supporting the spectral measure associated with the initial state imply lower bounds on a subsequence of time scales for the survival probability. This general phenomenon is illustrated with applications to the Fibonacci operator and the critical almost Mathieu operator. In particular, this gives the first quantitative dynamical bound for the critical almost Mathieu operator.

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1. Introduction

In this paper, we study solutions of the time-dependent Schrödinger equation

$$i\partial_t \psi(t) = H\psi(t),$$

where H is a bounded self-adjoint operator in a separable Hilbert space $\mathcal{H}$ and $\psi(0)=\psi\in\mathcal{H}$ with $\|\psi\|=1$. The case of main interest to us is when $\mathcal{H}=\ell^2(\mathbb{Z}^d)$ and H is a Schrödinger operator, $H=\Delta+V$. We denote the spectrum of H, which is a compact subset of $\mathbb{R}$, by $\sigma(H)$ and the spectral measure associated with the pair (H,ψ) by μ_ψ. The time evolution of the state ψ is given by $\psi(t)=\mathrm{e}^{-itH}\psi$.

A quantity of natural physical interest is the survival probability, that is, the probability of finding the state in the initial state at time t,

$$|\langle\psi(t),\psi(0)\rangle|^2 = \left|\langle \mathrm{e}^{-itH}\psi,\psi\rangle\right|^2 = \left|\int_{\sigma(H)} \mathrm{e}^{-itE}\, d\mu_\psi(E)\right|^2 = \left|\hat{\mu}_\psi(t)\right|^2.$$

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The second identity holds by the spectral theorem. If the spectrum of H is point spectrum, it follows from a theorem of Wiener, see ([10], Lemma 2.6, p. 54.), that the survival probability does not tend to *zero* as t goes to infinity. If the spectrum of H is absolutely continuous, then from the Riemann-Lebesgue Lemma it follows that the survival probability tends to *zero* ([10], p. 57).

An important result about the behavior of $\psi(t)$ in time average is the RAGE theorem, see [7]

THEOREM 1.

$$\lim_{t\to\infty}\frac{1}{T}\int_0^T\langle\psi,(t),A\psi(t)\rangle\mathrm{d}t=0$$

for any compact operator A if and only if μ_ψ *is purely continuous*

We refer the reader to Last [7] for the RAGE theorem and a general discussion of the survival probability and other natural ways of measuring how fast the state $\psi(t)$ changes with time.

To study the power-law decay of the survival probability, let us define

$$\gamma_\psi^{\pm}=\lim_{t\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\left|\hat{\mu}_\psi(t)\right|^{-2}}{\log t}.$$

It follows from the definition that

$$\gamma_\psi^-\leq\beta\quad\Leftrightarrow\quad\limsup_{t\to\infty}|\hat{\mu}_\psi(t)|^2t^{\beta+\varepsilon}=\infty\text{ for every }\varepsilon>0$$

and

$$\gamma_\psi^+\geq\beta\quad\Leftrightarrow\quad\liminf_{t\to\infty}|\hat{\mu}_\psi(t)|^2t^{\beta-\varepsilon}=0\text{ for every }\varepsilon>0.$$

It is well known, and discussed extensively in [7], that continuity properties of μ_ψ imply bounds for the *time-averaged* survival probability. Without the time-average, on the other hand, there are far fewer results; we refer the reader to [5,11] and references therein.

In [5] the authors consider the *pointwise* behavior of the Fourier transform of the spectral measure for discrete one-dimensional Schroedinger operators with sparse potentials. They find a resonance structure which admits a physical interpretation in terms of a simple quasiclassical model. In [11] the author studies, how small perturbations of the operator can affect the property of the survival probability being zero at infinity.

In this paper, we consider survival probabilities that are not time-averaged and employ a classical theorem of Salem [12] to deduce upper bounds for γ_ψ^- in cases where upper bounds for the Hausdorff dimensions of the support of the spectral measures are known.

2. Time Behavior of Survival Probabilities

Let us recall some basic facts about dimension of sets and measures. Given $S \subseteq \mathbb{R}$, a δ-cover is a covering of S by a countable set of intervals I_n of length at most δ. For $\beta \in [0, 1]$, the β-dimensional Hausdorff measure of S is

$$h^{\beta}(S) \equiv \lim_{\delta \to 0} \left[\inf_{\delta-\text{covers}} \sum_{n=1}^{\infty} | I_n |^{\beta} \right].$$

Notice that h^0 is the counting measure and h^1 the Lebesgue measure. For any S, there is a number $\beta_S \in [0, 1]$ such that $h^{\beta}(S) = 0$ if $\beta > \beta_S$, and $h^{\beta}(S) = \infty$ if $\beta < \beta_S$. This β_S is the Hausdorff dimension of S, denoted by $\dim_H S$.

DEFINITION 1. Let μ be a Borel measure on $\mathbb{R}$ and $\beta \in [0, 1]$.

(i) μ is called β-singular, denoted βs, if it is supported on a set S with $h^{\beta}(S) = 0$.
(ii) μ is called β-dimension singular, denoted βds, if it is supported on a set S with $\dim_H S \leq \beta$.

Here, we say that μ is supported on S if $\mu(\mathbb{R} \backslash S) = 0$. Note that if μ is βs, then it is βds, therefore the theorem below can be applied whenever our spectral measures are β-singular. We will discusse some cases in the next section.

In [12] R. Salem considered continuous monotonic fuctions which are singular and of the Cantor type, that is, which are constant in each interval contiguous to a perfect set of measure zero. This perfect set is called in [12] the spectrum of the function.

He proves the following result (thm III in[12]).

THEOREM 2. *No singular function (except constant) exists having as spectrum a perfect set of Hausdorff dimension* $\alpha > 0$, *and whose Fourier-Stieltjes transform belongs to* L^q *for some* $q < \frac{2}{\alpha}$

Likewise, no singular function (except constant) can have as spectrum a perfect set of Hausdorff dimension $\alpha > 0$, *and have Fourier Stieltjes coefficients of order* $n^{-\frac{\alpha}{2} - \varepsilon}, \varepsilon > 0$ *(no matter how small* ε *is).*

We shall reformulate the above theorem as follows:

THEOREM 3. *Let* $\psi \in \mathcal{H}$ *be a normalized vector and assume that* μ_{ψ} *is* βds *with* $\beta < 1$. *Then* $\gamma_{\psi}^{-} \leq \beta$. *That is,*

$$\limsup_{t \to \infty} |\hat{\mu}_{\psi}(t)|^2 t^{\beta+\varepsilon} = \infty \tag{1}$$

for every $\varepsilon > 0$. *Moreover,*

$$\hat{\mu}_{\psi}(t) \notin L^p(0, \infty) \text{ for every } p < \frac{2}{\beta}. \tag{2}$$

Proof. Essentially, this follows from [12, Theorem III]; see also [3, Sections 4.3 and 4.4], [4, Théorème III on p. 106], and [9, Chapter 12]. We give a sketch of the argument.

Fix a set S with $\dim_H S \leq \beta$ that supports μ_ψ. The Fourier transform of μ_ψ is a bounded continuous function which obeys

$$\hat{\mu}_\psi(-t) = \overline{\hat{\mu}_\psi(t)}. \tag{3}$$

Assume that (1) fails. Thus, observing (3), it follows that there is a constant C_1 with

$$|\hat{\mu}_\psi(t)| \leq C_1 |t|^{-\frac{\gamma}{2}} \tag{4}$$

for some $\gamma \in (\dim_H S, 1)$. Choose $\delta \in (\dim_H S, \gamma)$ and consider the δ-energy of μ_ψ, that is,

$$I_\delta(\mu_\psi) = \iint \frac{d\mu_\psi(x)\, d\mu_\psi(y)}{|x-y|^\delta}.$$

It follows from the Plancherel theorem that

$$I_\delta(\mu_\psi) = C_2 \int |t|^{\delta-1} |\hat{\mu}_\psi(t)|^2\, dt \tag{5}$$

for some suitable constant C_2; compare [9, Lemma 12.12].Here we use that H is bounded and therefore μ_ψ supported on a compact set

Thus, combining (4) and (5), we see that

$$\begin{aligned} I_\delta(\mu_\psi) &= C_2 \int_{|t|\leq 1} |t|^{\delta-1} |\hat{\mu}_\psi(t)|^2\, \mathrm{d}t + C_2 \int_{|t|>1} |t|^{\delta-1} |\hat{\mu}_\psi(t)|^2\, \mathrm{d}t \leq \\ &\leq C_2 \int_{|t|\leq 1} |t|^{\delta-1}\, \mathrm{d}t + C_1 C_2 \int_{|t|>1} |t|^{\delta-1} |t|^{-\gamma}\, \mathrm{d}t < \\ &< \infty \end{aligned}$$

since $\delta > 0$ and $\delta - \gamma < 0$. This shows that S has positive δ-capacity. By [3, Theorem 4.13], this implies $\dim_H S \geq \delta$. We obtain a contradiction because δ was chosen strictly larger than $\dim_H S$.

Now assume that (2) fails. Then we can choose $\nu > 0$ such that

$$\dim_H S + \nu < 1 \tag{6}$$

and

$$\hat{\mu}_\psi(t) \in L^p(\mathbb{R}) \quad \text{for } p = \frac{2}{\dim_H S + \nu}, \tag{7}$$

where we again used (3). Choose $\delta \in (\dim_H S, \dim_H S + \nu)$ and consider the δ-energy of μ_ψ. We have

$$\begin{aligned} I_\delta(\mu_\psi) &= C_2 \int_{|t|\le 1} |t|^{\delta-1} |\hat{\mu}_\psi(t)|^2 \,\mathrm{d}t + C_2 \int_{|t|>1} |t|^{\delta-1} |\hat{\mu}_\psi(t)|^2 \,\mathrm{d}t \le \\ &\le C_2 \int_{|t|\le 1} |t|^{\delta-1} \,\mathrm{d}t + \\ &\quad + C_2 \left(\int_{|t|>1} |t|^{\frac{\delta-1}{1-\dim_H S-\nu}} \,\mathrm{d}t \right)^{1-\dim_H S-\nu} \left(\int_{|t|>1} |\hat{\mu}_\psi(t)|^p \,\mathrm{d}t \right)^{\dim_H S+\nu} < \\ &< \infty. \end{aligned}$$

Here we used Hölder's inequality in the second step and the fact that

$$\frac{\delta-1}{1-\dim_H S-\nu} < -1,$$

which follows from our choice of δ, in the third step. As before, we may infer that S has positive δ-capacity and $\dim_H S \ge \delta$, which contradicts our choice of δ. This concludes the proof. □

Remark. If the measure μ_ψ is supported on a set of Hausdorff dimension 1, then the conclusions of Theorem 3 hold, provided μ_ψ has a non-trivial singular component. Indeed, if μ_ψ is not purely absolutely continuous, then $\hat{\mu}_\psi(t) \notin L^2(\mathbb{R})$; see [10, Corollary on p. 57]. This implies both (1) and (2) in this case.

3. Applications

We will first present applications of Theorem 3 to two quasi-periodic operators acting in $\ell^2(\mathbb{Z})$. In both cases, non-trivial upper bounds for the Hausdorff dimension of the spectrum are known. Since the spectrum supports all spectral measures, we can use it as the set S in question.

The first is the Fibonacci operator

$$[H_{\mathrm{Fib}}(\lambda,\theta)]\psi](n) = \psi(n+1) + \psi(n-1) + \lambda \chi_{[1-\alpha,1)}(n\alpha+\theta \mod 1)\psi(n),$$

where $\lambda > 0$, $\alpha = \frac{\sqrt{5}-1}{2}$ is the inverse of the golden ratio, and $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. It is easy to see that the spectrum of $H_{\mathrm{Fib}}(\lambda,\theta)$ does not depend on θ. It does, however, depend on λ, and it was shown in [1, Theorem 2] that for $\lambda \ge 8$, its Hausdorff dimension is bounded from above by $\frac{\log(1+\sqrt{2})}{\log\left[\frac{1}{2}\left((\lambda-4)+\sqrt{(\lambda-4)^2-12}\right)\right]}$. Thus, combining this result with Theorem 3 above, we obtain:

COROLLARY 1. *If $H = H_{\mathrm{Fib}}(\lambda, \theta)$ with $\lambda \geq 8$ and $\theta \in \mathbb{T}$ arbitrary, then we have for every initial state ψ,*

$$\gamma_\psi^- \leq \frac{\log(1+\sqrt{2})}{\log\left[\frac{1}{2}\left((\lambda-4)+\sqrt{(\lambda-4)^2-12}\right)\right]}$$

and

$$\hat{\mu}_\psi(t) \notin L^q(0,\infty) \text{ for every } q < \frac{2\log\left[\frac{1}{2}\left((\lambda-4)+\sqrt{(\lambda-4)^2-12}\right)\right]}{\log(1+\sqrt{2})}.$$

In particular, we see that γ_ψ^- can be made arbitrarily small by making λ sufficiently large. Some of the results from [1] have been generalized to other frequencies α by Liu et al. in [8]. Their results may be combined with Theorem 3 in an analogous way.

The second application of Theorem 3 we present involves the almost Mathieu operator

$$[H_{\mathrm{AMO}}(\lambda, \alpha, \theta)]\psi](n) = \psi(n+1) + \psi(n-1) + 2\lambda\cos(2\pi(n\alpha+\theta))\psi(n),$$

where $\lambda > 0$, $\alpha \in \mathbb{T}$ is irrational, and $\theta \in \mathbb{T}$. It is known that the Lebesgue measure of $\sigma(H_{\mathrm{AMO}}(\lambda, \alpha, \theta))$ is equal to $4|\lambda - 1|$ and hence it vanishes if and only if $\lambda = 1$. The Hausdorff dimension of the spectrum of $H_{\mathrm{AMO}}(1, \beta, \alpha)$ was studied by Last in [6]. From Theorem 3 above and [6, Theorem 2], we obtain:

COROLLARY 2. *Consider the almost Mathieu operator $H = H_{\mathrm{AMO}}(1, \alpha, \theta)$. If α is such that there are rational numbers $\frac{p_n}{q_n}$ with $q_n \to \infty$ and*

$$\lim_{n\to\infty} q_n^4 \left|\alpha - \frac{p_n}{q_n}\right| = 0,$$

and $\theta \in \mathbb{T}$ is arbitrary, then for every initial state ψ, $\gamma_\psi^- \leq \frac{1}{2}$ and $\hat{\mu}_\psi(t) \notin L^q(0,\infty)$ for every $q < 4$.

Remarks. (i) In addition to this explicit class of well-approximable α, there is a non-explicit class for which a stronger result can be shown. Indeed, a result in preparation by Last and Shamis for the almost Mathieu operator $H = H_{\mathrm{AMO}}(1, \alpha, \theta)$ will imply that there is a dense G_δ set of α's for which we have $\gamma_\psi^- = 0$ for every $\theta \in \mathbb{T}$ and every initial state ψ.

(ii) This result is of interest because quantum dynamical questions are notoriously difficult for the critical almost Mathieu operator, whereas the spectral type can be determined for all frequencies and all but countably many phases via duality and zero-measure spectrum.

Two other applications of Theorem 3 we only indicate briefly: In [2], del Rio et al. showed that local perturbations of the Anderson model in the localization regime (large coupling or suitable energy regions) have zero dimensional spectral measures. They even gave quantum dynamical consequences in terms of moments of the position operator. Theorem 3 above complements their result by providing results for survival probabilities. Finally, Zlatoš [13] exhibited Schrödinger operators with sparse potentials whose spectral measures are supported by sets of non-trivial Hausdorff dimension. Again, Theorem 3 applies and yields consequences for survival probabilities.

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