SEMINARIO DEL DEPARTAMENTO DE MÉTODOS MATEMÁTICOS Y NUMÉRICOS SOBRE OPERADORES DE SCHRÖDINGER Y TEMAS AFINES

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Ciudad Univesitaria, a 27 de agosto de 1999.

Escuché de labios de la filántropa Soumaya Domit, mujer que me pareció realmente admirable aunque apenas tuve la fortuna de conocerla, que las crisis son periodos de oportunidades y crecimiento. Sin crisis no se crece. Sin embargo, decía ella con dulcísima voz, lo que se hace en tiempos de crisis resulta más importante y trascendente de lo que se hace en otros momentos. Entre otros ejemplos Soumi, como le decían cariñosamente, hablaba de crisis personales, crisis de salud y crisis conyugales. Parece claro que en una crisis de salud la forma en que actuemos tendrá consecuencias de más peso que nuestro proceder en tiempos de bienestar. Como reaccionemos dentro de una crisis conyugal, puede destruir un matrimonio o convertirlo en algo más permanente y bello.

Recordando las palabras de Soumaya, volteo los ojos a lo que sucede en la Institución donde estudié y donde trabajo. Claramente la UNAM, una de las universidades más importantes del país, está viviendo un periodo de crisis. A las dificultades usuales para investigar, enseñar y divulgar la cultura, se suma el problema de un paro estudiantil ya muy prolongado.

Frente a esta situación algunos universitarios hemos decidido darnos con renovado vigor a nuestras actividades académicas. Como reacción a la agresión y al secuestro, hemos optado por el trabajo y el entusiasmo. De las actividades más estimulantes y agradables está el contacto con colegas que trabajan en temas afines. Es la intención de estas notas, recoger algunas de las conferencias que impartieron distinguidos investigadores visitantes durante el periodo del paro y que resultaron particularmente enriquecedoras.

Las visitas de los matemáticos invitados fueron posibles gracias al apoyo dado a través de Proyectos: (IN-102998 PAPIIT-UNAM), (27487E CONACyT), y de recursos canalizados al Departamento de Métodos Matemáticos y Numéricos del Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas.

Quiero agradecer a los científicos visitantes y a los patrocinadores, particularmente a la UNAM por su colaboración, que a pesar de la dificultad de la situación actual hicieron posible un ambiente intelectual excitante y vivo que durante largo tiempo ha caracterizado a la UNAM.

Rafael del Río.

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STARK EFFECT FOR ONE DIMENSIONAL SCHRODINGER OPERATOR SOME OPEN MATHEMATICAL PROBLEMS

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Abstract

This note summarizes a series of three lectures given at the department of Mathematics of the University of Mexico (UNAM) in May 1999. These lectures describe some recent results on one dimensional Stark operators. we also give some mathematical problems which are still open concerning their spectral analysis.

Key-Words: Schrödinger Operators, Stark Effect, Spectral Analysis.

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I.Introduction.

This note concerns the spectral analysis: results and open questions, of the family of (Schrö dinger) Stark operators defined on $L^2(\mathbf{R})$,

$$H(F) = -\frac{d^2}{dx^2} + V(x) + Fx; F < 0. \qquad \text{I.1}$$

This is a common fact that the Schrödinger equation associated to the hamiltonian $H(F)$, describes the forced propagation of an quantum particle (under the action of the electric field F) in some media, the interaction media-particle is given by the function potential V. We are particularly interested in the case of periodic or disordered media, accordingly V is a periodic or a random function, chosen (of course!) such that $H(F)$ is an observable i.e. an selfadjoint operator on $L^2(\mathbf{R})$. Two types of spectral questions are discussed below:

1- The analysis of the real spectrum.

Let $H_0(F) = -\frac{d^2}{dx^2}+Fx; F<0$ be the free Stark operator, this is an essentially selfadjoint operator on $C_0^\infty(\mathbf{R})$ (i.e. the set of complex functions with compact support on $\mathbf{R}$ and which are differentiable an infinite number of times) [Ka], moreover $H_0(F)$ is purely A-C (Absolutely Continuous) on $\mathbf{R}$ [Cy, Fro, Kir, Si]. The question which we want to study here is the stability of such a property under the perturbation V, this is a natural question and from a physical point of view, this can be read as, what kind of media is "apt to trap" the quantum particle in presence of the electric field.

2-The resonances.

The resonances are complex poles of the meromorphic continuation in an complex neighborhood of some real energy of the function

$$z \in \mathbf{C} \setminus \sigma(H(F)) \rightarrow \mathcal{F}_\psi(z) = (\psi, (H(F)-z)^{-1}\psi) \qquad \text{I.2}$$

for some $\psi \in L^2(\mathbf{R})$, see e.g. [Re,Si] for the interpretation of these poles in terms of scattering states. In the case of the Stark effect, the study of the spectrum of resonances is usually based on the following point of view, originally given in [La,Li], [Opp]. Let $H = -\frac{d^2}{dx^2} + V(x)$ be a selfadjoint operator on $L^2(\mathbf{R})$, it describes the stationary states of an quantum system and some of these states are stable under the dynamic associated to H. Then the question is, by adding the field, does this give rise to unstable states or resonances for the corresponding Stark operator $H(F) = H + Fx, F < 0$. Notice that, in the more general situation of random or periodic V which we want to study here, the point spectrum of H may be dense in $\mathbf{R}$ or empty! and the existence of stable states for H is not clear, so in that case we limit ourselves to the abstract definition I.2. This phenomena, if it exists for a given V, is called the Stark effect [La, Li].

II. The Spectral Analysis of $H(F)$

It is well known since many year [Be, Ca, Du, Si, So, We], that if $V \in C_b^s(\mathbf{R})$ for suitable s, (that is $V \in C^1(\mathbf{R})$ and has bounded derivatives up to the order s), then $H(F)$ is purely A-C on $\mathbf{R} \setminus \mathbf{D}$, where $\mathbf{D}$ is a some discrete set, some recent works (see e.g. [Ki] or [Sh]), show that the minimal condition requires for such a result is $s = 1 + 0$. We do not discuss this case here, however it must be noticed that the existence of eigenvalues embedded in the continuous spectrum (belonging to the discrete set $\mathbf{D}$) is not yet solved.

Notice also that the existence of A-C spectrum does not imply that the quantum particle is not affected by the potential V, since for such a system, the presence near the real axis of spectral resonances affect the propagation. We will discuss the existence of spectral resonances in the next section.

The case of singular V i.e. $V \notin C_b^s(\mathbf{R})$ is a more difficult question and many situations occur. An example of these facts can be seen in [Na, Pu], where the authors construct an C^∞-potential V, for which $H(F)$ is purely pure point, in that case V has not a bounded derivative. An other different situation is given when we consider potentials V which are piecewise differentiable and has some singularities periodically distributed on the real line. The case where the singularities are bounded jumps has been studied in [Br1], more precisely:

let $T > 0$, we denote by $\mathcal{C}_n = (nT, (n+1)T], n \in \mathbf{Z}$. We consider real and bounded potentials V on $\mathbf{R}$, such that, $V \in C_b^2(\mathcal{C}_n)$ and $sup_n\{V'_{\infty,n}\}, sup_n\{V''_{\infty,n}\} < \infty$, V_n being the restriction of the potential V on $\mathcal{C}_n$ and $V'_{\infty,n} = sup_{x\in\mathcal{C}_n}\{V'_n\}$, $V''_{\infty,n} = sup_{x\in\mathcal{C}_n}\{V''_n\}$.

Under these assumptions, the potential V has a derivative which is only defined in the distributional sense, hence V is "more singular" than the one given by the construction of [Na, Pu], but we have the following result:

theorem [Br1] : *there exists a set $\mathcal{F} \subset \mathbf{R}^-$, of full measure, such that for all $F \in \mathcal{F}$ the operator $H(F)$ is purely A-C on $\mathbf{R}$. Moreover $\sigma(H(F)) = \sigma_{ac}(H(F)) = \mathbf{R}$*

The proof of this theorem is based on the explicit construction of the generalized eigenfunction of the associated differential equation. We show that for each real energy E, there exits a fundamental set of solutions, $\{\phi_1, \phi_2\}$ satisfying, $\phi_i = o(x^{-1/4}), \phi_i' = o(x^{1/4}), i = 1,2$ as $x \rightarrow +\infty$, hence we are exactly as in the free case (i.e. $V = 0$). So by using some general criteria for the existence of A-C spectrum (see e.g. [Br, Mo] and the section IV below) we then prove the theorem.

It must be noticed that the assumption on the distribution of the jumps is strong. Hence if we consider instead, a distribution of jumps which is supported on a sequence of real numbers $\{x_n\}_{n\in Z}$ such that $x_n \rightarrow \infty$ and $|x_{n+1} - x_n| = o(n^{-\alpha})$, for some $\alpha > 0$ and small, our method fails. This is due to the fact that the solution grow and we can think that if α is small enough the growth is enough to create localized states and then point spectrum (see e.g. [Si, Las] for a mathematical description of this last statment). These results are also related to those of

[Ni] which shows if V is the gaussian white noise, then the spectrum is singular for all $F < 0$ with a spectral transition.

Along this line of examples, consider the so called Kroenig-Penney model i.e.,

$$V = \sum_{n \in Z} \alpha_n \delta(x-n) \tag{II.1}$$

where $\alpha_n, n \in N$ are real coupling constants. V is defined in the distributional sense and the Stark operator, $H(F) = -\frac{d^2}{dx^2} + V(x) + Fx$ is defined as the selfadjoint operator, $H(F) = -\frac{d^2}{dx^2} + Fx$ with domain,

$$\phi \in L^2(\mathbf{R}), \quad \phi, \phi' \in AC(\mathbf{R} \setminus \mathbf{Z}); \quad (-\frac{d^2}{dx^2} + Fx)\phi \in L^2(\mathbf{R} \setminus \mathbf{Z}) \quad \text{and}$$

$$\phi(n+0) = \phi(n-0) = \phi(n), \quad \phi'(n+0) - \phi'(n-0) = \alpha_n \phi(n), \quad n \in \mathbf{Z}, \tag{II.2}$$

[Shu,Sto], here the symbol AC means the set of absolute continuous functions. This model was considered by many authors (see e.g. [Shu,Sto] and references therein) and the most important result was given in [De, Si, So]. In this paper, the authors consider the corresponding Anderson model, i.e. $\{\alpha_n\}_{n\in Z}$ are independent and identically distributed (i.i.d) random variables. Under suitable conditions on the distribution of the α_n's they prove the existence of a spectral transition, more precisely,

theorem [De, Si, So] : *for almost every sequence of $\{\alpha_n\}_{n\in N}$ (with respect to the probability measure generated from the random variables α_n), the essential spectrum $\sigma_{ess}(H(F)) = R$, moreover there exists two constants $0 < F_1 < F_2 < \infty$ such that, if $0 < F < F_1$, $H(F)$ is pure point and if $F_2 < F$ the spectrum of $H(F)$ is purely continuous*

However this result does not characterize the continuous spectrum, for large value of intensity of the field. It is interesting to compare this last theorem with the one of [Br1] for potentials which are constant on each cell C_n and the amplitude of the jumps are just given by the α_n's, in that case the spectrum is purely continuous (A-C). This shows that the nature of the singularities plays here a very important role for the existence of point spectrum.

From these facts, a natural question is, what's happen in the periodic case? (i.e. $\alpha_n = \alpha$ for all integer n), this is actually a non solved problem and seems to be a question of interest for some models in solid states physics [Ao]. Contrary to the random situation, neither of the linear part and the periodic potential, in the Schrödinger equation give rise to the growth of the wave functions and then creates an L^2- state. From these simple arguments the spectrum of the corresponding Stark operator must be continuous (A-C). In fact this situation is not so clear, consider now a similar situation with the periodic potential,

$$V = \sum_{n \in Z} \alpha \delta'(x-n) \tag{II.3}$$

which is defined in the distributional sense, $H(F)$ is defined as in (II.2) with the following boundary conditions,

$$\phi'(n+0) = \phi'(n-0) = \phi'(n), \quad \phi(n+0) - \phi(n-0) = \alpha_n \phi'(n), \quad n \in \mathbf{Z}. \tag{II.2}$$

In this case we know some results on the spectrum of $H(F)$ (see e.g. [Av,Ex, La], [Ex], [As,Du, Ex]),

theorem: *Let $F, \alpha \neq 0$, then $\sigma_{ess}(H(F)) = R$, the A-C spectrum of $H(F)$ is empty and there exits a constant $F_0 > 0$ such that for $F < F_0$ there exits a set Ω of value of α for which $H(F)$ is purely pure point*

Combined these last partial results with the one obtained in [De, Si, So], this shows that the simple arguments evoked above fail, so the existence of point spectrum for the periodic Kroennig-Penney model remains an open question, particularly the well understanding of the mechanism which produces the growth of the eigensolution in this case.

III. The Stark Resonances

Our goal is now to study the spectrum of resonances of

$$H(F) = -\frac{d^2}{dx^2} + V(x) + Fx; F < 0. \tag{III.1}$$

defined on $L^2(\mathbf{R})$. Firstly, we describe a simple case which we call the "atomic case", where V consists in a regular negative function vanishing at infinity, see e.g. [Br2] for a complete mathematical description of this situation. In this case the spectrum of $H = -\frac{d^2}{dx^2} + V(x)$, $\sigma(H)$ is such that

$$\sigma(H) \cap (-\infty, 0] = \{E_1, E_2, E_3 ....\} \tag{III.2}$$

where each eigenvalue is associated to an L^2-state which is localized on the "well". Now consider such a state, let E_i be the corresponding eigenvalue, when the field is turned on, this localized state become an unstable state since now, the particle can go through the barrier by tunnel effect and escape away at infinity. This phenomena give rise to a resonance Z_i for $H(F)$. For small F, the real part ReZ_i is near E_i and the imaginary part satisfies the Landau-Oppenheimer law:

$$|ImZ_i| = o(exp\{-C_i/F\}) \tag{III.3}$$

for some constant C_i depending on the energy E_i. This implies that the life time of this unstable state [Re, Si], $\Gamma = o(exp\{C_i/F\})$ and clearly grows as F goes to zero. The rate C_i/F is the total rate of the exponential decay of the green function in the potential barrier at energy E_i. This rate is given by the size of the barrier in the Agmon sense and we have,

$$C_i/F = \int_{barrier} (V + Fx - E_i)^{1/2} dx. \tag{III.4}$$

Through this example, we see that the local decay of the green function is essential for the existence of the resonances.

We now want to consider the situation where V consists in a sum of atomic potentials,

$$V = \sum_{n \in Z} \alpha_n U_i(x-n) \tag{III.5}$$

where the function U_i are chosen such that V is well defined and regular. For this type of potential the situation is then more complex, since a priori we have no barrier such as in the atomic case.

Suppose that α_n are i.i.d. random variables, with an uniform distribution supported on $[\alpha_0, \alpha_1]$ $(0 < \alpha_0 < \alpha_1)$. It is well known [Ca, La] that almost surely, the solution of

$$-\frac{d^2\phi}{dx^2}(x) + V(x)\phi(x) = 0 \tag{III.6}$$

grow exponentially, and this is also true for any positive energy, so around the origin, saying in some interval $[-c/F, c/F]$, for a positive constant c and small enough, the linear perturbation is small of order c and then does not affect the behavior of the solution. This implies that in this region the green function exponentially decays, hence a state which is originally $(F=0)$ localized at the right of this barrier, give rise to unstable state when the field is turned on. Using these facts it is proven in [Be,Br], under suitable condition on the function $U_n, n \in \mathbf{Z}$,

theorem [Be, Br] : *For almost every sequence of $\{\alpha_n\}$, there exists $0 < F_0 < \infty$ such that, for any $0 < F < F_0$ H(F) has at least one resonance Z with a real part $ReZ \in \Delta$ and*

$$0 < |ImZ| \leq exp\{-\tau/F\} \tag{III.7}$$

for some positive constant τ depending on the sequence α_n

This theorem proves the existence of the resonances and gives an upper bound on the imaginary part, to show that it satisfies the Landau-Oppenheimer law, we need the following result of [As, Br]:

theorem [As, Br] : *Suppose that the potential V is analytic in a small strip around the real axis, suppose that $H(F)$ has a resonance, then there exists a constant τ' such that*

$$|ImZ| \geq exp\{-\tau'/F\} \tag{III.8}$$

Notice that this last result is more general than our case, in particular it can be applied for periodic potentials.

We turn on to some periodic models, for suppose that V is t-periodic and regular (analytic in some strip around the real axis), by applying the result of the section II, $H(F)$ is purely absolutely continuous on $\mathbf{R}$.

On the other hand, let T_t the unitary shift operator defined as, $\forall \psi \in L^2(\mathbf{R})$, $T_t\psi(x) = \psi(x+t)$, since $T_t H(F) T_t^{-1} = H(F) + Ft$, then if Z is a resonance for $H(F)$, the Stark-Wannier ladder defined by the set $L_Z = \{Z + nt; n \in \mathbf{Z}\}$ belongs to the spectrum of resonances.

The existence of the Stark-Wannier ladders has been studied by many authors, [Av], [Gr, Ma, Sa], [Bu, Di] and more recently in [Bu, Gri]. However these works concern only potential for which $H = -\frac{d^2}{dx^2} + V$ has a finite number of gaps, the case of general periodic potential is not yet solved.

An interesting approach which allows to us to understand the local exponential decay, was initiated in [Av] and [Be, Gr]. This can be described as follow: the reference energy $E = 0$, we denote by $(e_n, e_{n+1}), i = 1,2,3...$ the successive gap of the periodic operator, consider the intervals, $I_n = (e_n/F, e_{n+1}/F)$, let x_n the middle point of I_n, then for x near x_n, the solutions of the Schrödinger equation which can be written as:

$$-\frac{d^2\phi}{dx^2}(x) + V(x) + F(x - x_n)\phi(x) = -Fx_n\phi(x) \tag{III.9}$$

has an exponentially behavior because the effective energy $-Fx_n$, is in the gap of the periodic operator and the perturbation $F(x - x_n)$ is very small. This implies the exponential decay of the green function in these regions, hence as in the case evoked above, these regions play the role here of some effective barriers. If the total contribution of lenght gap is finite, then the resonances create by the presence of this barrier have a the life time $\Gamma = o(exp\{C/F\})$ for some positive and finite constant C.

It must be noticed that this discussion about resonances only concerns models involving analytic potentials, actually there is no general mathematical approach in the case of singular potential.

IV. Apriori estimates for Stark operators

The goal of this section is to give some estimates on the resolvent of the Stark operators, which are very useful for their spectral analysis, in particular under some conditions its imply the existence of A-C spectrum (see e.g. [Re, Si]). To do this, We use the method of [Br, Mo] which we sketch here. This method involves the following information on the asymptotic behavior of the generalized solutions, let E be an real energy and $F < 0$ for which there exists a fundamental set of solutions, $\{\varphi_1, \varphi_2\}$ of

$$H(F)\varphi = (-\frac{d^2}{dx^2} + V + Fx)\varphi = E\varphi \tag{IV.1}$$

such that

$$|\varphi_1\varphi_1' + \varphi_2\varphi_2'| < \infty \quad \text{and} \quad |\varphi_1^2 + \varphi_2^2| < \infty \quad \text{on} \quad I = (x_0, +\infty) \tag{IV.2}$$

for some $x_0 > 0$. Notice that by using the standard Liouville Green approximation [Ol], if V is bounded and smooth enough, this condition is satisfied for all real E and $F < 0$, it is shown in [Br1] that this is also true for V having jumps periodically distributed.
Let $\phi = (\varphi_1^2 + \varphi_2^2)^{1/2}$ and for $\varepsilon > 0$, $K_\epsilon = e^{\varepsilon \int_{x_0}^{x} (1-\chi)\phi^2 dt}$ where χ is a smooth characteristic function of I, denoting by $\langle x \rangle_\alpha = (1+|x|)^{-\alpha}, \alpha \in \mathbf{R}$, we have,

theorem [Br,Mo]: *let $-\infty < F < 0$ and and $E \in \mathbf{R}$ such that IV.2 is satisfied, for all $\alpha > 1/4$ and $u, v \in L^2(\mathbf{R})$, $||\phi u||, ||\phi v|| < \infty$, there exits a finite constant $c(E, F, \alpha, u, v)$ such that*

$$sup_{1>\varepsilon>0} \left| (u, \langle x \rangle_\alpha K_\varepsilon (H(F) - E - 2i\varepsilon)^{-1} K_\varepsilon \langle x \rangle_\alpha v) \right| \leq c(E, F, \alpha, u, v). \quad \text{(IV.3)}$$

Sketch of proof: Fix E, F as in the theorem and consider, for $\varepsilon > 0$, the approximate hamiltonian,

$$H(\varepsilon, F) = H(F) - 2\varepsilon(\phi^2)' + \varepsilon^2 \phi^4 \quad \text{(IV.4)}$$

which corresponds in some sense to the original operator in the limit $\varepsilon = 0$, moreover this hamiltonian has good properties which we give below. Notice that, for any $\varepsilon \in \mathbf{R}$, by (IV.2), $H(\varepsilon, F)$ is an essentially selfadjoint operator on $C_0^\infty(\mathbf{R})$. We have the following decomposition, valid in operator sense on $C_0^\infty(\mathbf{R})$,

$$H(\varepsilon, F) - E - 2i\varepsilon = (p - 1/\phi^2 - i\phi'/\phi - i\varepsilon\phi^2)(p + 1/\phi^2 + i\phi'/\phi + i\varepsilon\phi^2) \quad \text{(IV.5)}$$

where p denote the operator $i\frac{d}{dx}$. The identity (IV.5) is derived by a straightforward calculus, by noticing that $f = 1/\phi^2 + i\phi'/\phi$ satisfies the Riccati equation $if' - f^2 = V - E$ a.e. on $\mathbf{R}$. Define on $C_0^\infty(\mathbf{R})$,

$$\text{i)} A_+(\varepsilon) = (p + 1/\phi^2 + i\phi'/\phi + i\varepsilon\phi^2) = \phi a_+(\varepsilon)\phi^{-1}, a_+(\varepsilon) = p + \phi^{-2} + i\varepsilon\phi^2$$

$$\text{ii)} A_-(\varepsilon) = (p - 1/\phi^2 - i\phi'/\phi - i\varepsilon\phi^2) = \phi^{-1} a_-(\varepsilon)\phi, a_-(\varepsilon) = p - \phi^{-2} - i\varepsilon\phi^2, \quad \text{(IV.6)}$$

by standard arguments, the first order differential operators, $A_+(\varepsilon), A_-(\varepsilon)$ and $a_+(\varepsilon), a_-(\varepsilon)$ are closable on $L^2(\mathbf{R})$ (see e.g. [Ka]), we denote by the same symbol their closed extension. In the other hand, we define the inverse of $a_+(\varepsilon), a_-(\varepsilon)$ on the function $\psi \in L^2(\mathbf{R})$ with compact support by,

$$\text{i)} (a_+^{-1}(\varepsilon)u)(x) = e^{-\theta(x)} \int_{-\infty}^{x} e^{+\theta(t)} u(t) dt$$

$$\text{ii)} (a_-^{-1}(\varepsilon)u)(x) = e^{\theta(x)} \int_{x}^{-\infty} e^{-\theta(t)} u(t) dt \quad \text{(IV.7)}$$

where $\theta(x) = \varepsilon \int_{x_0}^{x} \phi^2 dt - i \int_{x_0}^{x} \phi^{-2}(t) dt$. By usual arguments of [Mo] (or directly from (IV.5)) the operators, $\phi a_+^{-1}(\varepsilon)\phi, \phi a_-^{-1}(\varepsilon)\phi$ are bounded operators, since $Im\{a_+(\varepsilon)\} = -Im\{a_-(\varepsilon)\} = \varepsilon\phi^2 \geq \varepsilon m_1^2 \phi^2$. These considerations together with the factorization (IV.5), give us a representation of the resolvent of $H(\varepsilon, F)$, for $\varepsilon > 0$, in terms of the resolvent of the first order differential operators, $a_+^{-1}(\varepsilon), a_-^{-1}(\varepsilon)$ by the formula,

$$(H(\varepsilon, F) - E - 2i\varepsilon)^{-1} = \phi a_+^{-1}(\varepsilon)\phi k - k\phi a_-^{-1}(\varepsilon)\phi \quad \text{(IV.8)}$$

where $k(x) = ie^{-2\theta(x)} \int_{-\infty}^{x} e^{2\theta(t)} \phi^{-2}(t) dt = -1/2 \left(1 - 2\varepsilon e^{-2\theta(x)} \int_{-\infty}^{x} \phi^2 e^{2\theta(t)} dt\right)$ and verifies $sup_x |k(x)| < 1$. Formulas (IV.7) and (IV.8) give us the necessary tools to analyze the boundary value as $\varepsilon \to 0$ of $(H(\varepsilon, F) - E - 2i\varepsilon)^{-1}$. By a direct calculation from (IV.7), we have the following estimates , for all $\alpha > 1/2$ and $(u, v) \in L^2(\mathbf{R}) \times L^2(\mathbf{R})$,

$$\sup_{\varepsilon > 0} \left| (u, \langle x \rangle_\alpha a_\pm^{-1}(\varepsilon) \langle x \rangle_\alpha v) \right| \leq \alpha^{-1} ||u||.||v||, \quad \text{(IV.9)}$$

hence by (IV.8), this implies,

$$\sup_{\varepsilon > 0} \left| (u \langle x \rangle_\alpha (H(\varepsilon, F) - E - 2i\varepsilon)^{-1} \langle x \rangle_\alpha v) \right| \leq 2\alpha^{-1} ||\phi u|| ||\phi v|| \quad \text{(IV.10)}$$

Then to prove the theorem we use the standard perturbation theory between the operators, $K_\varepsilon (H(\varepsilon, F) - E - 2i\varepsilon)^{-1} K_\varepsilon$ and $K_\varepsilon (H(F) - E - 2i\varepsilon)^{-1} K_\varepsilon$, the function K_ε decays exponentially and then control the region (a neighborhood of $-\infty$) where the solutions grow too much. We have

$$K_\varepsilon (H(\varepsilon, F) - E - 2i\varepsilon) - (H(F) - E - 2i\varepsilon) K_\varepsilon = \varepsilon W_\varepsilon - 2i\varepsilon(\chi - 1) A_+(\varepsilon) \quad \text{(IV.11)}$$

for some function W_ε which under our assumption, is bounded function and

$$\phi^2 A_+(\varepsilon)(H(\varepsilon, F) - E - 2i\varepsilon)^{-1} = \phi a_-^{-1}(\varepsilon)\phi \quad \text{(IV.12)}$$

valid in the bounded operator sense on $L^2(\mathbf{R})$, this identity follows from (IV.5) and (IV.8). Then by using (IV.9-11) together with the geometric resolvent equation and a quadratic estimate, we prove the theorem.

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Introduction to the spectral theory of Fibonacci-type operators

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Abstract

We provide an introduction to the spectral theory of Fibonacci operators and generalizations, namely, models generated by primitive substitutions and circle maps. In particular, we discuss spectral properties such as purely singular continuous spectrum and zero-measure Cantor spectrum for these operators. A criterion sufficient for both properties is presented. As an illustration we show that this criterion is obeyed by all Sturmian models. These notes arose from a lecture series given at a workshop on Schrödinger operators at IIMAS-UNAM, Mexico City, Mexico in May 1999.

1 Introduction

Our concern is the study of discrete one-dimensional Schrödinger operators in $l^2(\mathbb{Z})$, that is,

$$(H\phi)(n) = \phi(n+1) + \phi(n-1) + V(n)\phi(n), \tag{1}$$

where the arrangement of potential values $V(n)$, $n \in \mathbb{Z}$, represents a structure intermediate between order and disorder. The primary example we have in mind is the Fibonacci operator. This model was proposed in 1983 by Kohmoto et al. [19] and Ostlund et al. [23]. These authors exhibited a scaling structure which led them to study an associated dynamical system. They observed that the eigenfunctions are critical in the sense that they are neither extended nor localized. Kohmoto and Oono conjectured in [20] that the spectrum is purely singular continuous and nowhere dense. Soon after the discovery of quasicrystals by Shechtman et al. in 1984 [25], the Fibonacci model became the standard example of a one-dimensional quasicrystal. The rigorous investigation of the spectral properties started only a few years later. Works by Casdagli [7], Sütő [26, 27], and Bellissard et al. [4] confirmed the claims of the early works which partially relied on numerical investigations and proved that the spectrum is indeed purely singular continuous and a Cantor set of zero Lebesgue measure. Models generalizing crucial properties of the Fibonacci operator received considerable attention in the following years, we refer the reader to [1, 2, 3, 5, 6, 9, 10, 11, 12, 15, 16, 17, 18] for some important contributions. It became clear that the above spectral picture is shared by the generalized models. A very good review of the state of affairs as of 1994 together with an introduction to the necessary background from spectral theory has been given in a Les Houches winter school course by Sütő [28].

It is the purpose of these notes to present some of the methods that have been successfully applied to Fibonacci-type operators and to explain how the somewhat unusual spectral properties naturally arise in these models. The organization is as follows. In Section 2 we describe two ways of looking at the Fibonacci sequence. Consequently, Section 3 introduces two classes of sequences, both being extensions of a specific property of the Fibonacci sequence. Moreover, the associated Schrödinger operators are defined. General results from the theory of ergodic families of Schrödinger operators are presented in Section 4 including a profound lemma by Kotani [21] which is extremely useful in the case of Fibonacci-type operators. Namely, building upon this lemma we present in Section 5 a unified treatment of both the spectral type and the Lebesgue measure of the spectrum. Finally, we show in Section 6 how to verify the necessary input of this general approach in the case of Sturmian models, the latter being the most prominent extension of the Fibonacci operator since these models share almost all its crucial properties. The material presented in Sections 5 and 6 has been obtained in collaboration with D. Lenz and will be published in a more elaborate form in [14].

2 The Fibonacci sequence and two generation schemes

This section is concerned with a brief discussion of the Fibonacci sequence with emphasis on two generation schemes. A finite patch in this sequence, in fact a prefix, is given by the following arrangement of 0's and 1's,

$$1011010110110101101011011010110 1$$

We shall discuss a generation process using iteration of a substitution rule and a direct and pointwise definition.

1. A substitution process: Consider the symbols 0 and 1 and define on this set of symbols the substitution (or replacement) rule

$$\begin{aligned} 0 &\to 1, \\ 1 &\to 10. \end{aligned}$$

Now start with the symbol 1, apply this rule, and apply it successively to the patches obtained, where the replacement works in a symbol-by-symbol fashion,

$$\begin{aligned} 1 &\to 10 \\ &\to 101 \\ &\to 10110 \\ &\to 10110101 \\ &\to 1011010110110 \\ &\vdots \end{aligned}$$

Two key observations can be made. Any iterate is a prefix of its successor, that is, by passing to the next iterate one only has to add some symbols. In fact, one may observe (and easily prove) that any iterate is obtained by concatenating the two preceding iterates. From this, the second key observation follows, namely, that the number of symbols in the sequence of iterates follows a Fibonacci rule. Now, from the prefix property it follows that there is a unique infinite sequence having all the iterates as prefixes. This sequence is called *Fibonacci sequence* and is denoted by v_F.

2. A circle map: Let $\alpha_g = \frac{\sqrt{5}-1}{2}$ be the golden mean and define $v_F' : \mathbb{N} \to \{0,1\}$ by

$$v_F'(n) = \chi_{(1-\alpha_g,1]}(n\alpha_g \mod 1).$$

Both mechanisms yield a two-valued one-sided sequence and it can in fact be shown that

$$v_F = v_F'. \tag{2}$$

This property, however, is non-trivial (but well known) and the proof involves number theory. This sequence gives rise to a *hull* which is simply the set of two-sided sequences that locally look like the Fibonacci sequence, that is,

$$\Omega = \{\omega \in \{0,1\}^{\mathbb{Z}} : \text{every finite patch in } \omega \text{ occurs in } v_F\}.$$

It is our objective here to discuss the spectral properties of discrete one-dimensional Schrödinger operators with potentials from the hull Ω or generalizations thereof. Even though the model appears to be very simple, the spectral theory of these operators is far from being trivial. On the contrary, rather spectacular phenomena such as nowhere dense spectrum and purely singular continuous spectral measures are the rule for these models. Our goal is to convince the reader that these phenomena occur naturally and we will even present large parts of the proofs, supplemented by appropriate references to the original literature.

3 Primitive substitution sequences, sequences generated by circle maps, and associated Schrödinger operators

We have seen that the Fibonacci sequence and the induced hull naturally belong to two classes. It is the purpose of this section to introduce and study these two classes.

Primitive substitutions: Let $A = \{a_1, \dots, a_r\}$ be a finite set. Denote by A^* the set of words over A, that is, $a_{i(1)} \dots a_{i(l)}$, $i(j) \in \{1, \dots, r\}$. A mapping $S : A \to A^*$ is called *substitution*. This mapping can be extended morphically to A^* and $A^{\mathbb{N}}$. A fixed point $u \in A^{\mathbb{N}}$ of S is called *substitution sequence*. In the case that there exists $a \in A$ such that $S(a)$ begins with a and the length of $S^n(a)$ tends to infinity as $n \to \infty$, $u = S^\infty(a)$ exists and is a substitution sequence. As an example consider the Fibonacci substitution, $A = \{0,1\}$, $S(0) = 1$, $S(1) = 10$. We can choose $a = 1$ and $u = S^\infty(a)$ reads $1011010110110\dots$, which is nothing but our friend, the Fibonacci sequence. A substitution is called *primitive* if there exists $k \in \mathbb{N}$ such that for every $i,j \in \{1,\dots,r\}$, $S^k(a_i)$ contains a_j. Obviously, the Fibonacci substitution is pimitive, choose $k = 2$. The reader can readily check that given a primitive substitution S, S^p has a fixed point for a suitable power $p \in \mathbb{N}$. Having found a substitution sequence u, one may extend this sequence to the left arbitrarily and denote this doubly infinite sequence by $\tilde{u}$.

Circle maps: Choose $\alpha \in (0,1)$ irrational and $\beta \in (0,1)$ arbitrary. Define over $A = \{0,1\}$ the two-sided sequence $u_{\alpha,\beta}$ by

$$u_{\alpha,\beta}(n) = \chi_{[1-\beta,1)}(n\alpha \mod 1).$$

In case $\alpha = \beta$, the model is called Sturmian.

On the set $A^{\mathbb{Z}}$ of two-sided sequences over A we have the *shift* $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by $(Tv)_n = v_{n+1}$. Given a two-sided sequence v over some A (e.g., $\tilde{u}$ or $u_{\alpha,\beta}$), we define a closed and T-invariant subset Ω of $A^{\mathbb{Z}}$ (i.e., a *subshift*) by

$$\Omega = \{\omega \in A^{\mathbb{Z}} : \omega = \lim_{i\to\infty} T^{n_i} v, n_i \to \infty\}.$$

We could also define the hull using the subword condition from above (using the subwords occurring in the one-sided sequence obtained by restricting v to $\mathbb{N}$). The reader may check that these definitions are equivalent. It is known and in fact not hard to see that the definition of Ω does not depend on the choice of the substitution sequence u in the primitive substitution case. So far we have obtained a topological dynamical system (Ω, T). Let us now discuss the existence of invariant and ergodic measures on the Borel σ-algebra of Ω, where we have discrete topology on A and product topology on $A^{\mathbb{Z}}$. A good reference is the book by Queffélec [24]. The Borel σ-algebra of Ω is generated by so-called *cylinder sets*, that is, sets of the form

$$[b_1 \dots b_l]_j = \{\omega \in \Omega : \omega(j) \dots \omega(j+l-1) = b_1 \dots b_l\}, \tag{3}$$

where $b_1 \dots b_l$ is a subword of v. On the other hand, given a subword $b_1 \dots b_l$ of v, we define its *frequency* in v by

$$d_v(b_1 \dots b_l) = \lim_{n\to\infty} \frac{1}{n} \#\{j \leq n : v(j) \dots v(j+l-1) = b_1 \dots b_l\}, \tag{4}$$

provided that the limit exists. It is an interesting connection and in fact a standard result from the theory of topological dynamical systems that they are related for the models under consideration in the following way.

Proposition 3.1 *Suppose $d_v(b_1 \dots b_l)$ exists and is strictly positive for every subword $b_1 \dots b_l$ of v. Then there is a unique ergodic T-invariant Borel measure μ on Ω. Moreover, we have*

$$\mu([b_1 \dots b_l]_j) = d_v(b_1 \dots b_l) \tag{5}$$

for every $j \in \mathbb{Z}$.

Remark. A Borel measure ν on Ω is called T-invariant if $\nu(T^{-1}B) = \nu(B)$ for every Borel set B. A T-invariant measure ν is called ergodic if $T^{-1}B = B$ for some Borel set B implies $\nu(B) = 0$ or $\nu(B) = 1$.

Proposition 3.2 *Suppose v is generated by a primitive substitution or by a circle map. Then, all frequencies exist and are strictly positive.*

Thus, in these cases we have a unique measure μ such that the system (Ω, T, μ) is ergodic. Let $f : A \to \mathbb{R}$ be a non-constant function and define for $\omega \in \Omega$ the potential $V_\omega : \mathbb{Z} \to \mathbb{R}$ by $V_\omega(n) = f(\omega_n)$. This gives rise to an ergodic family of Schrödinger operators H_ω, $\omega \in \Omega$, where

$$(H_\omega \phi)(n) = \phi(n+1) = \phi(n-1) + V_\omega(n)\phi(n).$$

4 Eigenfunctions, the Lyapunov exponent, and Kotani's theorem

In this section we present some basic methods which are often used in the study of spectral properties of one-dimensional Schrödinger operators. Moreover, we discuss some topics related to ergodic families and recall the fundamental Kotani theory.

Consider a discrete one-dimensional Schrödinger operator of the form (1) in $l^2(\mathbb{Z})$, where the potential V will be assumed bounded here for simplicity. One is interested in identifying the sets $\sigma(H)$, $\sigma_{\rm pp}(H)$, $\sigma_{\rm sc}(H)$, and $\sigma_{\rm ac}(H)$. Among the deepest and most useful results in the general theory are those that relate these sets to solutions of the eigenvalue equation in sequence sense, namely,

$$\phi(n+1) + \phi(n-1) + V(n)\phi(n) = E\phi(n), \tag{6}$$

where $E \in \mathbb{C}$. An obvious relation is given by

$$E \text{ eigenvalue of } H \Leftrightarrow (6) \text{ has an } l^2\text{-solution},$$

and $\sigma_{\rm pp}(H)$ is then the closure of the set of these energies E. Interestingly, also the other parts of the spectrum can be characterized in terms of solutions to (6), by results of Schnol, Berezanskii, Gilbert-Pearson, and Last-Simon. To study the solutions it is convenient to rewrite (6) in the following way. Define

$$\Phi(n) = \begin{pmatrix} \phi(n+1) \\ \phi(n) \end{pmatrix},$$

$$T_E(n) = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix},$$

$$M_E(n) = \begin{cases} T(n) \times \cdots \times T(1) & n \geq 1 \\ I & n = 0 \\ T(n+1)^{-1} \times \cdots \times T(0)^{-1} & n \leq -1 \end{cases}$$

Then one may easily check that

$$\phi \text{ solves } (6) \Leftrightarrow \Phi \text{ solves } \Phi(n) = M_E(n)\Phi(0) \text{ for every } n \in \mathbb{Z}.$$

The matrices $M_E(\cdot)$ are called *transfer matrices*. The linear space of solutions to (6) for fixed E is two-dimensional, as can be seen from the above relation. Consider the two solutions $\phi_{1,2}$ induced by the initial conditions $\phi_1(0) = \phi_2(1) = 0$, $\phi_1(1) = \phi_2(0) = 1$ which are obviously linearly independent and therefore form a basis of this two-dimensional space. Then we also have

$$M_E(n) = \begin{pmatrix} \phi_1(n+1) & \phi_2(n+1) \\ \phi_1(n) & \phi_2(n) \end{pmatrix}.$$

Thus, the matrix contains all information about $\phi_{1,2}$ and hence all solutions in a pointwise way. In particular, bounds on $||M_E(n)||$, for example, yield bounds on $||\Phi(n)||$ for all solutions.

Let us now consider ergodic families of discrete one-dimensional Schrödinger operators, compare, for example, the book by Cycon et al. [8]. To this end, let (Ω, T, μ) be ergodic and $g : \Omega \to \mathbb{R}$ measurable. Define potentials $V_\omega : \mathbb{Z} \to \mathbb{R}$ by $V_\omega(n) = g(T^n\omega)$ (in our two examples above we considered the special case of a subshift (Ω, T) with unique ergodic measure μ and $g(\omega) = f(\omega_0)$ with some non-constant $f : A \to \mathbb{R}$). The associated operators H_ω are given by

$$(H_\omega\phi)(n) = \phi(n+1) + \phi(n-1) + V_\omega\phi(n).$$

The spectral properties of such a family of operators are deterministic in the sense that there exist sets $\Omega_0 \subseteq \Omega$, $\Sigma, \Sigma_{\rm pp}, \Sigma_{\rm sc}, \Sigma_{\rm ac} \subseteq \mathbb{R}$ such that $\mu(\Omega_0) = 1$ and $\sigma(H_\omega) = \Sigma, \sigma_{\rm pp}(H_\omega) = \Sigma_{\rm pp}, \sigma_{\rm sc}(H_\omega) = \Sigma_{\rm sc}, \sigma_{\rm ac}(H_\omega) = \Sigma_{\rm ac}$ for every $\omega \in \Omega_0$. It follows from [22] that in the case of families generated by a primitive substitution or by a circle map, $\sigma(H_\omega) = \Sigma, \sigma_{\rm ac}(H_\omega) = \Sigma_{\rm ac}$ even hold for every $\omega \in \Omega$. Index the associated transfer matrices by ω (i.e., $T_{E,\omega}(n)$, $M_{E,\omega}(n)$). It turns out that when studying the growth of $||M_{E,\omega}(n)||$, it is useful to distinguish between exponential growth and subexponential growth. Thus, define

$$\gamma_\omega(E) = \limsup_{n\to\infty} \frac{1}{n} \ln ||M_{E,\omega}(n)||.$$

Theorem 4.1 (Furstenberg-Kesten) *For every $E \in \mathbb{C}$ there exist $\Omega_E \subseteq \Omega$ and $\gamma(E) \in \mathbb{R}$ such that $\mu(\Omega_E) = 1$ and for every $\omega \in \Omega_E$, the* lim sup *in the definition of $\gamma_\omega(E)$ is a limit and this limit equals $\gamma(E)$.*

For certain models this can even be strengthened. In the primitive substitution case it is known that $\Omega_E = \Omega$ for every $E \in \mathbb{C}$ [16]. In the Sturmian case one has $\Omega_E = \Omega$ for every $E \in \Sigma$ [13]. The number $\gamma(E)$ is called *Lyapunov exponent*. The beautiful and very deep Kotani theory now establishes a link between the Lyapunov exponent and the absolutely continuous spectrum. Define

$$A = \{E \in \mathbb{R} : \gamma(E) = 0\}.$$

The essential support of a set $S \subseteq \mathbb{R}$ is defined by

$$\overline{S}^{ess} = \{E \in \mathbb{R} : |(E-\varepsilon, E+\varepsilon) \cap S| > 0 \, \forall \varepsilon > 0\},$$

where $|\cdot|$ denotes Lebesgue measure. In particular, $\overline{S}^{ess} = \emptyset$ for every set S of zero Lebesgue measure. The following two theorems are among the most useful results in the theory of ergodic one-dimensional Schrödinger operators.

Theorem 4.2 (Kotani) $\Sigma_{ac} = \overline{A}^{ess}$.

Theorem 4.3 (Kotani) *If the potentials V_ω are aperiodic and take only finitely many values, then $|A| = 0$. In particular, $\Sigma_{ac} = \emptyset$.*

Another useful result is the following.

Theorem 4.4 (Osceledec) *Suppose that for some $E \in \mathbb{C}$, $\gamma(E) > 0$. Then, for every $\omega \in \Omega_E$ there exist solutions ϕ_d^+, ϕ_d^- of $H_\omega \phi = E\phi$ such that ϕ_d^+ (resp., ϕ_d^-) decays exponentially at $+\infty$ (resp., $-\infty$) at the rate $-\gamma(E)$. Moreover, every solution which is linearly independent of ϕ_d^+ (resp., ϕ_d^-) grows exponentially at $+\infty$ (resp., $-\infty$) at the rate $\gamma(E)$.*

Thus, in the case of positive Lyapunov exponent one has a complete understanding of the asymptotics of the solutions at infinity.

5 A unified treatment of the basic problems using half-line eigenfunction estimates

Let us consider an ergodic family $(H_\omega)_{\omega \in \Omega}$ such that the potentials V_ω are aperiodic and take only finitely many values. This class comprises the families generated by (non-trivial) primitive substitutions and by circle maps. By Kotani we know $|A| = 0$ and hence $\Sigma_{ac} = \emptyset$. By Last-Simon, we even know that the absolutely continuous spectrum is empty for all ω for the two subclasses of particular interest. On the other hand, no example is known such that either $|\Sigma| > 0$ or $\Sigma_{pp} \neq \emptyset$. The goal is thus to establish purely singular continuous zero-measure spectrum for as many families in this class as possible. In this section we shall describe a unified treatment. Interestingly, this approach was initiated by a study of stability aspects, namely, whether local perturbations can introduce eigenvalues inside the spectrum. We shall exhibit a simple criterion which is sufficient for answering this question in the negative and which also proves purely singular continuous zero-measure spectrum.

Suppose that for some $\omega \in \Omega$, the operator H_ω has empty point spectrum. Then, for every $E \in \mathbb{C}$, the equation

$$\phi(n+1) + \phi(n-1) + V_\omega(n)\phi(n) = E\phi(n), \tag{7}$$

does not admit an l^2-solution. We ask whether this property is preserved under local perturbations. To this end, consider a finitely supported $W : \mathbb{Z} \to \mathbb{R}$ and the induced perturbed eigenvalue equation

$$\phi(n+1) + \phi(n-1) + (V_\omega(n) + W(n))\phi(n) = E\phi(n). \tag{8}$$

Suppose that (7) has solutions ϕ_+, ϕ_- such that ϕ_+ (resp., ϕ_-) is square-summable at $+\infty$ (resp., $-\infty$). Then, one may find a suitable perturbation W such that (8) has an l^2-solution and hence $E \in \sigma_{pp}(H_\omega + W)$. Note that this situation occurs for all energies E which are outside the original spectrum, that is, $E \notin \sigma(H_\omega)$. It is therefore natural to consider energies from the original spectrum when studying stability of absence of l^2-solutions. Let us define the following stability set,

$$\Omega_s = \{\omega \in \Omega : \text{for every } E \in \Sigma, \text{ (7) has no non-trivial solution which is square-summable at } +\infty\}.$$

Note that Σ equals the spectrum of H_ω for almost every $\omega \in \Omega$, and since we are interested in properties that hold almost surely, we can always ignore nasty sets of measure zero. In the two cases of interest we always have uniformly constant spectrum anyway as noted above, so that this complication does not occur in these cases. Now for $\omega \in \Omega_s$, a local perturbation W cannot introduce an eigenvalue inside Σ since it will still be true that for energies $E \in \Sigma$, no solution of the perturbed eigenvalue equation (8) is square-summable at $+\infty$. Of course, one could as well work with the left half-line or make an E-dependent choice of half-line. The following result shows that a study of the stability set can not only give purely singular continuous spectrum but also zero-measure spectrum!

Theorem 5.1 *Consider an ergodic family of discrete one-dimensional Schrödinger operators such that the potentials V_ω are aperiodic and take only finitely many values. Suppose $\mu(\Omega_s) > 0$. Then we have*

1. *$\Sigma_{pp} = \emptyset$. More precisely, for almost every $\omega \in \Omega$, we have that the operator H_ω has purely singular continuous spectrum and for every finitely supported $W : \mathbb{Z} \to \mathbb{R}$, the operator $H_\omega + W$ has no eigenvalues inside Σ.*
2. *$|\Sigma| = 0$.*

Remark. Although the stability condition ensures that for $\omega \in \Omega_s$ and $W : \mathbb{Z} \to \mathbb{R}$ finitely supported, the operator $H_\omega + W$ has no eigenvalues inside Σ, it is in general not true that $\sigma_{pp}(H_\omega + W) \cap \Sigma = \emptyset$ since the perturbation W may introduce discrete spectrum in the gaps of Σ which has accumulation points in Σ.

Proof. It is clear from the definition that Ω_s is a T-invariant set. It follows from the assumption $\mu(\Omega_s) > 0$ that we have $\mu(\Omega_s) = 1$ which implies all the properties claimed in part 1. The second part follows by Kotani's theorem once we show $\Sigma \subseteq A$. Let $E \in \Sigma$. We will show $\gamma(E) = 0$. Assume to the contrary $\gamma(E) > 0$. By Furstenberg-Kesten there exists $\Omega_E \subseteq \Omega$ such that $\mu(\Omega_E) = 1$ and for every $\omega \in \Omega_E$, the norm of $M_{E,\omega}(n)$ grows exponentially in n at the rate $\gamma(E)$. Now pick some $\omega \in \Omega_s \cap \Omega_E$, which is of course non-empty. By Osceledec this implies the existence of a solution to (7) which decays exponentially at $+\infty$, thus contradicting $\omega \in \Omega_s$. □

We learn that a proof of $\mu(\Omega_s) > 0$ has very nice implications. It is therefore desirable to find a criterion for this condition which can be checked by standard methods. To present such a criterion, let us define for $n \in \mathbb{N}$ and $C < \infty$ the following sets which are measurable since they are cylinder sets (check!),

$$G(n, C) = \{\omega \in \Omega \;:\; V_\omega(k) = V_\omega(k+n), 1 \le k \le n, \; |\mathrm{tr}(M_{E,\omega}(n))| \le C \;\; \forall E \in \Sigma\}.$$

Proposition 5.2 *Suppose there is $C < \infty$ such that $\limsup_{n\to\infty} \mu(G(n,C)) > 0$. Then, $\mu(\Omega_s) > 0$.*

Proof. Without loss of generality we may assume $C \ge 1$. By assumption, we have $\mu(\limsup(G(n,C))) > 0$. The assertion follows if we show

$$\limsup(G(n, C)) \subseteq \Omega_s. \tag{9}$$

Consider an arbitrary $\omega \in \limsup(G(n,C))$. Then, for a suitable sequence $n_k \to \infty$, ω belongs to $G(n_k, C)$. We therefore have for every $k \in \mathbb{N}$ and every $E \in \Sigma$, $M_{E,\omega}(2n_k) = (M_{E,\omega}(n_k))^2$ and $|\mathrm{tr}(M_{E,\omega}(n_k))| \le C$. Together with

$$(M_{E,\omega}(n_k))^2 - \mathrm{tr}(M_{E,\omega}(n_k)) M_{E,\omega}(n_k) + I = 0$$

this implies for every solution ϕ of (7),

$$\Phi(2n_k) - \mathrm{tr}(M_{E,\omega}(n_k))\Phi(n_k) + \Phi(0) = 0.$$

This in turn yields

$$\max(\|\Phi(2n_k)\|, |\mathrm{tr}(M_{E,\omega}(n_k))| \cdot \|\Phi(n_k)\|) \geq \tfrac{1}{2}\|\Phi(0)\|$$

and hence by $C \geq 1$,

$$\max(\|\Phi(2n_k)\|, \|\Phi(n_k)\|) \geq \frac{1}{2C}\|\Phi(0)\|$$

for every k. In particular, $\Phi(n) \not\to 0$ as $n \to \infty$ if ϕ is non-trivial. Thus we obtain $\omega \in \Omega_s$ and hence the assertion (9). □

6 The stability set in the Sturmian case

In this section we prove that the stability set has positive measure in the Sturmian case, thus presenting an example of the applicability of the general theorem from the previous section. At some points, however, we will point to appropriate references for further details as the proof requires some non-trivial facts about Sturmian models.

So fix some irrational $\alpha \in (0,1)$, write u_α for $u_{\alpha,\alpha}$, and consider the generated subshift (Ω, T). Recall that the definition of the associated operators $(H_\omega)_{\omega\in\Omega}$ depends on the function $f : \{0,1\} \to \mathbb{R}$. Define $\lambda = |f(0) - f(1)|$. In order to prove that $\mu(\Omega_s) > 0$ we want to show that the assumption of Proposition 5.2 is satisfied, that is, we want to study bounds on transfer matrix traces and frequencies of squares in u_α. This can be done in three steps. First, one exhibits recursive structures in u_α. Second, one translates these recursive structures to the level of transfer matrices and obtains a recursion on a certain sequence of scales. By passing to the traces of these matrices one gets a (generalized) dynamical system, the *trace map*. The spectrum can then be characterized in terms of the behavior of orbits under the trace map. Finally, one estimates the frequencies of appropriate squares.

6.1 Recursive structures in u_α

Consider the continued fraction expansion of α, that is,

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{10}$$

with uniquely determined $a_n \in \mathbb{N}$. The associated rational approximants $\frac{p_n}{q_n}$ obey

$$p_0 = 0,\ p_1 = 1,\ \ p_n = a_n p_{n-1} + p_{n-2}, \tag{11}$$

$$q_0 = 1,\ q_1 = a_1,\ \ q_n = a_n q_{n-1} + q_{n-2}. \tag{12}$$

Define the words s_n over $A = \{0,1\}$ by

$$s_n = u_\alpha(1)\dots u_\alpha(q_n). \tag{13}$$

In particular, the length of the word s_n is equal to q_n, $n \geq 0$. Bellissard et al. have shown in [4] that for every $n \geq 3$, we have the following analog to (12) on the level of words which in a way extends (2),

$$s_n = s_{n-1}^{a_n} s_{n-2}. \tag{14}$$

They employed the fact that the continued fraction approximants $(\frac{p_n}{q_n})_{n\in\mathbb{N}}$ are best approximants to α in the sense that for every n, no rational number $\frac{p}{q}$ with $q < q_{n+1}$ is closer to α than $\frac{p_n}{q_n}$. For later use, we also want to note the following elementary formula,

$$s_n s_{n+1} = s_{n+1} s_{n-1}^{a_n - 1} s_{n-2} s_{n-1},\ \ n \geq 3, \tag{15}$$

which can easily be verified using the above recursive relation.

6.2 The trace map and uniform bounds for energies inside the spectrum

The next step is to find an analog of (14) on the level of transfer matrices. Define for $E \in \mathbb{C}$ and $n \in \mathbb{N}$ the following matrix which is essentially the transfer matrix over some word s_n,

$$M_{E,n} = \begin{pmatrix} E - f(u_\alpha(q_n)) & -1 \\ 1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} E - f(u_\alpha(1)) & -1 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

We immediately deduce the following recursion for these matrices,

$$M_{E,n} = M_{E,n-2} M_{E,n-1}^{a_n}. \tag{17}$$

Define $x_E(n) = \mathrm{tr}(M_{E,n})$. It is now possible to deduce a recursion for these traces from the recursion (17). However, this is not as straightforward as the above steps. One has to use that every $M_{E,n}$ obeys

$$M_{E,n}^2 = x_E(n) M_{E,n} - I \tag{18}$$

since $\det(M_{E,n}) = 1$. Using this, some calculation shows that $x_E(n)$ depends in a recursive way on $x_E(n-1), x_E(n-2)$, and $x_E(n-3)$, see [4] for the explicit form of this *trace map*. A very nice and useful fact is now the following characterization of energies from the spectrum. Let $C_\lambda = 2 + \sqrt{\lambda^2 + 8}$. Then [4],

$$E \in \Sigma \text{ if and only if } |x_E(n)| \leq C_\lambda \text{ for every } n \in \mathbb{N}.$$

By invariance of the trace under cyclic permutation of factors, these bounds hold as well for cyclic permutations of the product in (16), provided $E \in \Sigma$. This suggests that Proposition 5.2 may be applicable with $C = C_\lambda$.

6.3 Frequencies of squares and applicability of Proposition 5.2

Following the suggestion from above, we consider $\limsup_{n\to\infty} \mu(G(n, C_\lambda))$. By the trace map characterization of the spectrum, we have $G(q_n, C_\lambda) \supseteq G(n)$, where

$$G(n) = \{\omega \in \Omega : \omega_1 \dots \omega_{2q_n} \sim s_n^2\}. \tag{19}$$

Here, $\omega_1 \dots \omega_{2q_n} \sim s_n^2$ means that $\omega_1 \dots \omega_{2q_n}$ is a cyclic permutation of s_n^2. Note that this implies that $\omega_1 \dots \omega_{q_n}$ is a cyclic permutation of s_n. In order to establish $\mu(\Omega_s) > 0$ it is now sufficient to show

$$\limsup_{n\to\infty} \mu(G(n)) > 0. \tag{20}$$

Recall that the measure μ of a cylinder set is given by the frequency of the defining word, compare equation (5). Using this and a cyclicity argument (s_n^3 contains all the cyclic permutations of s_n^2), one gets

$$\mu(G(n)) \geq q_n \cdot d_{u_\alpha}(s_n^3). \tag{21}$$

The frequency of s_n^3 in u_α can now be estimated using (13) and (14). In fact, these equations show that every subword of u_α having length at least $7q_n$ contains a copy of s_n^3. To see this, partition u_α on $\mathbb{N}$ into blocks of type s_n and blocks of type s_{n-1}. From (14) we infer that in this partition, blocks of type s_{n-1} are always isolated and blocks of type s_n always have multiplicity a_{n+1} or $a_{n+1}+1$. This and (15) proves the above claim. Thus, $d_{u_\alpha}(s_n^3) \geq \frac{1}{7q_n}$ and hence

$$\limsup_{n\to\infty} \mu(G(n, C_\lambda)) \geq \limsup_{n\to\infty} \mu(G(n)) \geq \frac{1}{7} > 0. \tag{22}$$

This proves that in the Sturmian case, the stability set has positive measure. Our general result Theorem 5.1 is therefore applicable in this case. Let us remark that based upon the method developed in [12], it can even be shown that in the Sturmian case, we always have $\Omega_s = \Omega$.

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GAP ESTIMATES OF THE SPECTRUM OF THE ZAKHAROV-SHABAT SYSTEM

B. Grébert

Abstract

We prove new gap estimates for the Zakharov–Shabat systems with complex periodic potentials. Our method allow us to characterize in a precise way the decreasing properties of the gap length sequence in terms of the regularity of the complex potentials.

We consider the Zakharov–Shabat operator (see [6])

$$L(\varphi,\psi) := i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{d}{dx} + \begin{pmatrix}0 & \varphi\\ \psi & 0\end{pmatrix}$$

with φ, ψ complex valued periodic functions of period 1 in the weighted Sobolev space

$$H_\omega := \left\{ f \in L^2(S^1, \mathbf{C}), \|f\|_\omega < \infty \right\}$$

where

$$\|f\|_\omega := \left(\sum_{k\in\mathbf{Z}} \omega(k)^2 |\hat{f}(k)|^2 \right)^{\frac{1}{2}},$$

$\hat{f}(k)(k \in \mathbf{Z})$ denote the Fourier coefficients of f and $\omega = (\omega(k))_{k\in\mathbf{Z}}, \omega(k) \geq 1$, denotes a symmetric submultiplicative weight, i.e.

$$\omega(k) = \omega(-k), k \in \mathbf{Z},$$

$$\omega(k+j) \leq \omega(k)\omega(j)\ ,\ k,j \in \mathbf{Z}.$$

As on example of submultiplicative weight we mention Abel-Sobolev weight, $\omega_{a,b}(k) := (1+|k|)^a\ e^{b|k|}$ with $a \geq 0$ and $b \geq 0$. An element $f \in H_{\omega_{a,b}}$ can be viewed as a complex valued function $f(z) = \sum_{k\in\mathbf{Z}} \hat{f}(k)e^{2i\pi kz}, z = x + iy$, analytic in the strip $|y| < \frac{b}{2\pi}$ and such that $f(x + \frac{ib}{2\pi})$ as well as $f\left(x - \frac{ib}{2\pi}\right)$ are in the Sobolev space $H^a(\mathbf{R}, \mathbf{C})$.

For our main Theorem we will need the following additional assumption on ω:

(A) there exists $0 < \delta < \frac{1}{2}$ such that the weight w^* defined by $(k \in \mathbf{Z})\omega^*(k) := \frac{\omega(k)}{(1+|k|)^\sigma}$ is still a submultiplicative weight.

Notice that in the case of Abel-Sobolev weight $\omega_{a,b}$, (A) is satisfied if $(a, b) \neq (0, 0)$.

Denote $\sigma(\varphi, \psi)$ the set of periodic eigenvalues of $L(\varphi, \psi)$ considered on the interval [0,2]:

$$\begin{aligned} \sigma(\varphi,\psi) : &= \{\lambda \in \mathbf{C}/\exists F \in H^1_{loc}(\mathbf{R}, \mathbf{C}^2), F \not\equiv 0, \\ L(\varphi,\psi)F &= \lambda F \text{ and } F(x+2) = F(x), x \in \mathbf{R}\}. \end{aligned}$$

Notice that $L(\varphi, \psi)$ is selfadjoint if and only if $\psi = \bar{\varphi}$ and in this case $L(\varphi, \bar{\varphi})$ is unitary equivalent to the well known AKNS operator (see [1]). Actually there already exist many results concerning the selfadjoint case (see [2], [5]). In the general case, $\sigma(\varphi, \psi)$ is discrete and using Rouché Theorem (see [3]) we verify that we can ordered the eigenvalues (enumerated with their algebraic multiplicities) in such a way that $\sigma(\varphi, \psi) = \{\lambda_k^-, k \in \mathbf{Z}\} \cup \{\lambda_k^+, k \in \mathbf{Z}\}$ with $Re\lambda_k^- \leq Re\lambda_k^+ \leq Re\lambda_{k+1}^-$ and $\lambda_k^\pm \underset{|k|\to\infty}{\sim} k\pi$. In particular for k large enough, the eigenvalues come in pairs $\{\lambda_k^+, \lambda_k^-\}$, i.e λ_k^+ and λ_k^- are close to each other and separated of rest of $\sigma(\varphi, \psi)$ by a distance of size $\frac{\pi}{2}$.

In [3] we prove the following

Theorem 1 *Let ω be a symmetric submultiplicative weight satisfying (A) and let γ be a constant such that $0 < \gamma < \delta$.*

Then for any bounded subset B in $H_\omega \times H_\omega$ there exists $M \geq 1$ so that the eigenvalues satisfy the following estimates

(i) $\sum_{k\in\mathbf{Z}} \omega(k)^2 |\lambda_k^+ - \lambda_k^-|^2 \leq M,$

(ii) $\sum_{k\in\mathbf{Z}} (1 + |k|)^{2\gamma} \omega(k)^2 |\lambda_k^+ - \lambda_k^- - 2(\hat{\psi}(k)\hat{\varphi}(-k))^{\frac{1}{2}}|^2 \leq M.$

In the selfadjoint case $\psi = \bar{\varphi}$) **(ii)** can be improved

$$\sum_{k\in\mathbf{Z}} (1 + |k|)^{4\gamma} \omega(k)^2 (\lambda_k^+ - \lambda_k^- - 2|\hat{\varphi}(k)|)^2 \leq M.$$

In particular our Theorem is a generalization of the gap estimates established by Marčenko [5] who considered only the selfadjoint case assuming $\omega = \omega_{a,0}$ with $a \in \mathbf{N}\backslash\{0\}$ (see also [2] for the case $\omega = \omega_{0,0}$ not contained in our Theorem). The proof uses a method developped in [4] for the Hill's equation.

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MAXIMAL PROPERTIES OF THE NORMALIZED CAUCHY TRANSFORM

ALEXEI POLTORATSKI

ABSTRACT. We study the normalized Cauchy transform in the unit disk. Our goal is to find analogs of classical results on Cauchy transforms and nontangential maximal operators for the case of arbitrary weights.

Let μ be a positive finite measure on the unit circle of the complex plane and $f \in L^1(\mu)$. Denote by $K\mu$ and $Kf\mu$ Cauchy integrals of measures μ and $f\mu$ correspondingly. The normalized Cauchy transform is defined as $C_\mu : f \mapsto \frac{Kf\mu}{K\mu}$. We prove that C_μ is bounded as an operator in $L^p(\mu)$ for $1 < p \leq 2$, has weak type (1,1) and unbounded (in general) for $p > 2$. The associated maximal nontangential operator is bounded for $1 < p < 2$, has weak types $(1,1)$ and $(2,2)$ and unbounded for $p > 2$.

1. INTRODUCTION.

This note is devoted to one of the classical objects of complex analysis, Cauchy integral.

For any summable function f on the unit circle $\mathbb{T}$ one can define its Cauchy integral Kf in the unit disk $\mathbb{D}$:

$$Kf(z) = \int_{\mathbb{T}} \frac{f(\xi)dm(\xi)}{1-\bar{\xi}z}, \tag{1}$$

where m is the normalized Lebesgue measure on $\mathbb{T}$. By the Fatou Theorem, Kf can also be defined on $\mathbb{T}$ by its non tangential boundary values. After such an extension one can view Cauchy integral as a "transform", i. e. an operator in $L^p(m)$ which sends f into the boundary values of Kf.

A classical theorem by M. Riesz says that Cauchy transform is bounded in L^p when $1 < p < \infty$. Another fundamental result by A. Kolmogorov states that the operator has weak type (1,1), which means that for any $f \in L^1(m)$ function Kf belongs to $L^{1,\infty}(m)$, where $L^{1,\infty}(m)$ denotes "weak L^1":

$$L^{1,\infty}(m) = \{f | \sup_{t>0} m(\{|f| > t\}) < \frac{C}{t},\ \ C < \infty\}.$$

Further progress is due to R. Hunt, B. Mackenhoupt and R. Wheeden who studied spaces with absolutely continuous weights $L^p(w)$. It was shown that Cauchy transform is bounded

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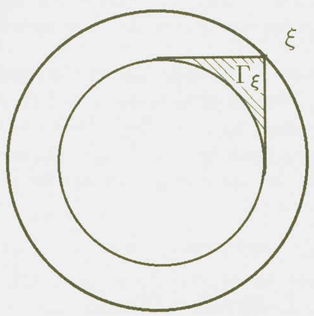
in $L^p(w)$, $1 < p < \infty$ or has weak type $(1,1)$ for $p = 1$ if and only if the weight w satisfies the celebrated A_p-condition.

Another classical object of complex function theory is the non tangential maximal function. For any function g in $\mathbb{D}$ we can define its non tangential maximal function Mg as

$$Mg(\xi) = \sup_{z \in \Gamma_\xi} |g(z)|$$

where $\xi \in \mathbb{T}$ and Γ_ξ is the sector $\{|z - \xi| \leq \sqrt{2}\,\mathrm{Re}(1 - \bar{\xi}z), |z| > \frac{1}{\sqrt{2}}\}$ (see Pic. 1 below).



Pic. 1

If f is from $L^p(m)$, or from a weighted space $L^p(w)$, one can consider its Cauchy integral Kf and the maximal function MKf. After that one can, once again, view MK as a transform in L^p. It is well known that in this regard MK has exactly the same properties as K: it is bounded in $L^p(m)$, $1 < p < \infty$ and has weak type (1,1). Moreover, the analogy extends to the case of absolutely continuous weights: MK is bounded in $L^p(w)$, $1 < p < \infty$ iff w satisfies A_p and has weak type (1,1) iff w satisfies A_1.

We refer the reader to [G] for the bibliography related to the above discussion.

These fundamental results on Cauchy transform and the non tangential maximal function play a prominent role in many areas of complex function theory. They give useful estimates on the behavior of Cauchy integrals in a complex domain. Some of the most important applications of Cauchy integrals lie in operator theory and mathematical physics. In these areas a Cauchy integral can appear as a resolvent function of an operator or, for instance, as a Weyl function related to a Schrödinger equation. In such applications, however, the corresponding weight (spectral measure) is rarely absolutely continuous, which makes the classical results insufficient. The purpose of this paper is to find an analog of the classical theory for the case of arbitrary weights.

Let $M(\mathbb{T})$ be the space of finite complex measures on the unit circle $\mathbb{T}$. We denote by $M_+(\mathbb{T})$ the subset consisting of positive measures. Let $\mu \in M_+(\mathbb{T})$ and $f \in L^p(\mu)$. Now, if we want to study Kf as in the classical theory, we have a slight problem: if, for example, μ is a singular measure then f may not be defined almost everywhere on $\mathbb{T}$, which makes the integration with respect to the Lebesgue measure in (1) impossible. The natural way

out is to integrate with respect to μ, i. e. to consider $Kf\mu$:

$$Kf\mu = \int_{\mathbb{T}} \frac{f(\xi)d\mu(\xi)}{1 - \bar{\xi}z}.$$

Now another question arises. If μ is, once again, singular then the boundary values of $Kf\mu$ can be infinite μ-almost everywhere. To correct that we have to normalize our Cauchy transform and consider $C_\mu f$:

$$C_\mu f = \frac{Kf\mu}{K\mu} = \frac{\int_{\mathbb{T}} \frac{f(\xi)d\mu(\xi)}{1-\bar{\xi}z}}{\int_{\mathbb{T}} \frac{d\mu(\xi)}{1-\bar{\xi}z}}. \tag{2}$$

In comparison with (1), we first include the measure μ in $Kf\mu$, but then "factor it out" dividing by $K\mu$. Now we can ask all the classical questions about the action of the normalized Cauchy transform C_μ and associated maximal operator MC_μ in $L^p(\mu)$. These questions are the focus of the present paper.

The function theoretical aspects of the normalized Cauchy transform were studied by D. Clark [C], D. Sarason [S], A. Aleksandrov [A1-5] and the author [P1-3]. Operator C_μ also appears in a number of applications, including problems in operator theory, mathematical physics and perturbation theory, see for instance [C] or [P4-6].

First, we would like to present a brief survey of some known properties of this operator.

Let us look at the image of $L^2(\mu)$ under C_μ. If $\mu = m$ then $C_\mu(L^2(\mu))$ is the standard Hardy space H^2. If μ is a singular measure, we obtain the famous model space $K^2_\theta = H^2 \ominus \theta H^2$ where θ is an inner function related to μ by the formula

$$K\mu = \frac{1}{1 - \theta}, \tag{3}$$

see [C]. Spaces K^2_θ play crucial role in operator model theory as the only invariant subspaces of the backward shift operator, see [N]. Finally, when μ is an arbitrary measure having nontrivial absolutely continuous and singular parts $C_\mu(L^2(\mu))$ gives us yet another well known space, the de Branges-Rovnyak space $\mathcal{M}^2_\theta$, where θ again satisfies (3), but now is not inner, see [S].

Here we will mostly concern ourselves with the second of these three examples. When μ is singular C_μ not only sends $L^2(\mu)$ onto K^2_θ, it actually preserves the norms [C] (the norm in K^2_θ is inherited from H^2). This beautiful fact can be viewed as an analog of Plancherel's formula and C_μ itself as a generalization of the Fourier transform.

For any μ C_μ is bounded as an operator from $L^2(\mu)$ to H^2 (or to $L^2(m)$). The situation becomes more interesting for $p \neq 2$:

Theorem 1 [A2]. *For any $\mu \in M_+(\mathbb{T})$ operator C_μ is bounded as an operator from L^p to H^p (to $L^p(m)$) for $1 < p \leq 2$ and has weak type (1,1). Operator C_μ is unbounded, in general, for $p > 2$. In particular, if μ is singular and $C_\mu : L^p(\mu) \mapsto H^p$ ($L^p(m)$) is bounded for some $p > 2$ then μ is discrete.*

In Section 3 we show that if $C_\mu : L^p(\mu) \mapsto H^p$ is bounded for some $p > 2$ for a singular μ then $\mathrm{supp}\,\mu$ can not contain an arc, see Example 4. Hence boundedness happens only in degenerate cases when μ is a pure point measure supported on a zero Cantor set.

Now we can return to our main goal which is to study C_μ and MC_μ as transforms in $L^p(\mu)$. Let us start with C_μ. First, we must establish the mechanism of the action of C_μ. As in classical case $C_\mu f$ is an analytic function in $\mathbb{D}$ which has non tangential boundary values a. e. on $\mathbb{T}$ with respect to the Lebesgue measure. But that does not mean that $C_\mu f$ has boundary values μ-a. e., since μ is now allowed to have nontrivial singular part. The following theorem takes care of this problem.

Theorem 2 [P1]. *Let $\mu \in M(\mathbb{T})$ and $f \in L^1(\mu)$. Then function $C_\mu f$ has finite non tangential boundary values μ-a. e. These values are equal to f μ^s-a. e., where μ^s is the singular component of μ.*

Now we know that C_μ can be naturally defined as a transform in $L^p(\mu)$. Moreover, the problem of its boundedness is solved for all $p \geq 1$ in the most difficult case of singular μ. For such measures C_μ is not only bounded, it is *identical.* This reflects the fact that for singular μ the measure $f\mu$ is never "antianalytic."

For arbitrary μ the situation is not as nice. The same "forces" that make Theorem 1 fail for $p > 2$ play a role here. All in all, we have the following

Theorem 3. *For any $\mu \in M_+(\mathbb{T})$ the operator C_μ is bounded in $L^p(\mu)$ for $1 < p \leq 2$ and has weak type $(1,1)$.*

As we will see from Example 1, the existence of measures such that C_μ is unbounded for $p > 2$ follows directly from Theorem 1. We will prove this theorem in Section 3.

Now let us discuss the action of the maximal operator $MC_\mu f$ in $L^p(\mu)$. This part contains the main new result of the paper.

From Theorem 2 we know that $MC_\mu f$ is at least finite μ-a.e. As usual, we say that an operator has weak type (p,p) if it acts from L^p into "weak L^p ", defined as

$$L^{p,\infty}(m) = \{f \mid \sup_{t>0} m(\{|f| > t\}) < \frac{C}{t^p},\ C < \infty\}$$

(with the inf of such C raised into the power $1/p$ viewed as a norm).

As we will discuss in Section 4, one of the results of [P1] implies that MC_μ has weak type (2,2). For a while that was all that was known about this maximal operator. Now we can present the following result.

Theorem 4. *For any $\mu \in M_+(\mathbb{T})$ the maximal operator MC_μ is bounded in $L^p(\mu)$ for $1 < p < 2$. It has weak type (1,1) and (2,2).*

As to the case $p > 2$, in Section 3 we will construct an example of singular μ and $f \in L^\infty(\mu)$ such that $MC_\mu f \notin L^p(\mu)$ for any $p > 2$. Hence the only remaining question is weather MC_μ has a strong type (2,2) (it can not have strong type (1,1): just put $\mu = m$). It seems that the answer should be negative, but a counterexample is yet to be found.

In classical case, as we saw in the beginning of the paper, the maximal operator always follows the pattern of the Cauchy transform itself: they are bounded under the same conditions. With the normalized Cauchy transform the situation is different. As we discussed above, if we restrict our attention to the case of singular measures μ, then the boundedness

of C_μ will trivially follow from Theorem 2 for *all* $p \geq 1$. The maximal operator, however, will still be, generally, unbounded for $p > 2$, as we will see from Example 2.

The weak type (1,1) for MC_μ can be generalized into a slightly stronger statement also discussed in Section 4:

Theorem 5. *If $\mu, \nu \in M(\mathbb{T})$, $\mu > 0$ then*

$$M\frac{K\nu}{K\mu} \in L^{1,\infty}(\mu).$$

A few finishing remarks. When trying to prove Theorems 2–5 the first natural idea is to apply classical estimates for analytic functions inside the disk such as the Schwartz Lemma, Harnak's Lemma, the maximal theorems mentioned in the beginning or Carleson's Theorem on the embedding of Hardy spaces. One quickly realizes however that all these results are based on the direct estimates of Poisson kernel. Hence if such a prove could be found it would also work for the operator

$$f \mapsto M\frac{Kf\mu}{P\mu}$$

where $P\mu$ is Poisson integral of μ.

This observation brings up a question that seems interesting by itself: what can be said about this operator? In particular, does $M\frac{K\nu}{P\mu}$ belong to $L^{1,\infty}(\mu)$ as in Theorem 5? Example 3 shows that the answer is negative: there exist $\mu, \nu \in M(\mathbb{T})$, $\mu > 0$ such that

$$M\frac{K\nu}{P\mu} \notin L^q(\mu)$$

for any $q > 0$. This illustrates the fact that our problem concerns singular integrals and can not be treated with "nonsingular" methods.

After that realization, the next natural idea is to apply classical "singular" tools such as Calderon-Zygmund theory. The following objection immediately arises. If one replaces the Cauchy transform K in the definition of our operator C_μ with the standard Hilbert transform H, understood in the sense of principal value, then Theorems 2–5 will no longer be valid. The easiest counterexample is $\mu = m + \delta_1$, where δ_1 denotes the unit point mass at 1, and f defined as 1 on the upper half-circle and as 0 elsewhere. Then $C_\mu f$ is infinite at point 1, which has a nonzero measure. At first sight this objection may not seem serious. We may try, for instance, to consider only continuous measures μ (without point masses). But new counterexamples can be found for this situation as well.

This difficulty in application of the classical "singular" theory reflects the fact that our results describe a slightly different phenomena. Theorems 2–5 hold true not because the singular part of Cauchy transform (conjugate Poisson integral) is "good" μ-a. e., but because at μ-a. e. point it is either "good" or "small" in comparison with nonsingular part (Poisson integral). The nonsingular part, in its turn, satisfies all the desired conditions and much more. Hilbert transform as well as any other "purely singular" operator can not be substituted into our theorems because it does not have such a nonsingular part.

In the next section we introduce our main tools and make preparations for the proofs. In Section 3 we give all the announced examples. Section 4 contains the proofs.

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2. Weighted level sets of inner functions.

In the introduction we defined "weak L^p" spaces $L^{p,\infty}(\mathbb{T})$ for $\infty > p > 0$. The corresponding "weak Hardy spaces" can now be defined as

$$H^{p,\infty} = L^{p,\infty}(\mathbb{T}) \cap H^q, \ q < p$$

with the $L^{p,\infty}(\mathbb{T})$ norm.

We also introduced space K^2_θ corresponding to an inner function θ as an orthogonal difference

$$K^2_\theta = H^2 \ominus \theta H^2.$$

Space K^2_θ is a Hilbert space. It possesses a system of reproducing kernels k_λ with the property that for any $\lambda \in \mathbb{D}$ and any $f \in K^2_\theta$ $f(\lambda) = < f, k_\lambda >$. The kernels are defined as

$$k_\lambda(z) = \frac{1 - \theta(\bar{\lambda})\theta(z)}{1 - \bar{\lambda} z}.$$

Spaces K^p_θ as well as weak spaces $K^{p,\infty}(\mathbb{T})$ can now be defined as closed spans of k_λ's in the topology of the corresponding Hardy spaces.

Each function from the space K^p_θ $(K^{p,\infty}_\theta)$ is known to possess the so-called "pseudo-continuation." Furthermore, these spaces are the only closed invariant subspaces of the backward shift operator $S^*: H^p \mapsto H^p$, $(S^*f)(z) = (f(z) - f(0))/z$, see [N].

A necessary and sufficient condition for a function f to belong to K^p_θ $(K^{p,\infty}_\theta)$ is that function $\tilde{f}$ defined on $\mathbb{T}$ as $\bar{f}\theta$ belongs to H^p $(H^{p,\infty})$ and $\tilde{f}(0) = 0$. It is easy to check that if $f(0) = 0$ then $\tilde{f}$ also belongs to K^p_θ $(K^{p,\infty}_\theta)$. The operation $f \mapsto \tilde{f}$ is a natural convolution on the subspace of K^p_θ $(K^{p,\infty}_\theta)$ consisting of functions f satisfying $f(0) = 0$. This subspace can be shown to have codimension 1.

We will call $f \in K^p_\theta$ $(f \in K^{p,\infty}_\theta)$ a Hermitian element if $f = \tilde{f}$. Every Hermitian element f satisfies

$$\frac{f}{\bar{f}} = \theta \tag{4}$$

a. e. on $\mathbb{T}$.

Let $\theta \in H^\infty$, $||\theta||_\infty \le 1$. Then for every $\alpha \in \mathbb{T}$ function $(\alpha + \theta)(\alpha - \theta)^{-1}$ has positive real part in $\mathbb{D}$. Therefore there is a unique measure σ_α in $M_+(\mathbb{T})$ such that

$$P(\sigma_\alpha) = \int_{\mathbb{T}} \frac{(1 - |z|^2) d\sigma_\alpha(\xi)}{|z - \xi|^2} = \text{Re}\,\frac{\alpha + \theta}{\alpha - \theta}.$$

In this way we can relate a family of measures $M_\theta = \{\sigma_\alpha\}_{\alpha \in \mathbb{T}}$ to any function θ from the unit ball of H^∞.

Conversely, for each measure σ in $M_+(\mathbb{T})$ there exists $\theta \in H^\infty$, $||\theta||_\infty \le 1$ such that $\sigma = \sigma_1 \in M_\theta$. Thus, the correspondence $\theta \leftrightarrow \sigma_1$ (as well as any correspondence $\theta \leftrightarrow \sigma_\alpha$, $\alpha \in \mathbb{T}$) provides a parameterization of the unit ball of H^∞ by means of measures from $M_+(\mathbb{T})$.

Under this parametrization the subset of singular measures corresponds to the set of all inner functions. Indeed, suppose θ is inner. Then $(\alpha + \theta)(\alpha - \theta)^{-1}$ is pure imaginary a. e. on $\mathbb{T}$ and therefore all measures σ_α are singular. It is not hard to see that the measure σ_α is supported on the set of points where the non tangential limits of θ are equal to α. It follows that the measures σ_α are not only singular with respect to Lebesgue measure, but also pairwise singular.

In the case of inner θ one can regard measures σ_α as weighted level sets $\{\theta = \alpha\}$ of θ. The weight reflects the speed of convergence of θ to its boundary values at different points on $\mathbb{T}$. In particular, σ_α has a point mass at $\xi \in \mathbb{D}$ iff $\theta(\xi) = \alpha$ and θ has a non tangential derivative at ξ, i. e. $\frac{\theta - \alpha}{z - \xi}$ has a finite non tangential limit at ξ (see [S]).

As we discussed in the introduction, for any inner θ

$$K^2_\theta = C_{\sigma_1}(L^2(\sigma_1)).$$

In this formula σ_1 can be replaced with any σ_α. Theorem 2 now says that any $f \in K^2_\theta$ has non tangential boundary values σ_α-a. e. for any $\alpha \in \mathbb{T}$ which belong to $L^2(\sigma_\alpha)$. Moreover, f can be recovered from its boundary values via C_{σ_α}.

It is not difficult to notice that if f is a Hermitian element then its boundary values satisfy (4) not only a. e. but also σ_α-a. e. on $\mathbb{T}$ for every $\alpha \in \mathbb{T}$. Since $\theta = \alpha$ σ_α-a. e., equation (4) transforms into

$$\frac{f}{\bar{f}}(\xi) = \alpha \tag{5}$$

for σ_α-a. e. ξ. In particular, in Section 4 we will use the fact that any Hermitian element is real σ_1-a. e. and imaginary σ_{-1}-a. e. Conversely, if for some $\alpha \in \mathbb{T}$ f satisfies (5) σ_α-a. e. and $f(0) = \int f d\sigma_\alpha = 0$, then f is a hermitian element.

Here are a few more simple observations that we will need in our proofs. We denote by P_+ the standard Riesz projection.

Lemma 6. *If $f \in K^2_\theta$ then:*

1) $f^2 \in K^1_{\theta^2}$.

2) The functions $v = P_+(\bar{\theta} f^2)$ and $u = f^2 - \theta v$ belong to $K^{1,\infty}_\theta$.

3) Function f^2 satisfies $f^2 = u + \theta v$.

4) Let $M_\theta = \{\mu_\alpha\}_{\alpha \in \mathbb{T}}$. Then for any $\alpha \in \mathbb{T}$

$$C_{\mu_\alpha} f^2 = u + \alpha v. \tag{6}$$

5) Let $\theta(0) = 0$. Then all measures μ_α are probability measures. If $M_{\theta^2} = \{\sigma_\alpha\}_{\alpha \in \mathbb{T}}$ then σ_α's can be derived from μ_α's through the following formula:

$$\sigma_\alpha = 1/2(\mu_{\sqrt{\alpha}} + \mu_{-\sqrt{\alpha}}). \tag{7}$$

Proof.

1) Notice that function $\bar{f}^2\theta^2$ belongs to H^1 and has value 0 at 0.

2) $u, v \in H^{1,\infty}$ by the standard properties of the Riesz projector. Now the statement follows from the fact that $\bar{\theta}u$ and $\bar{\theta}v$ are antianalytic.

3) Can be verified by simple calculations.

4) If f is a bounded function then u, v and $C_{\mu_\alpha}f^2$ all belong to K^2_θ. Moreover, $C_{\mu_\alpha}f^2$ and $u+\alpha v$ have the same boundary values μ_α-a. e. because $f^2 = u + \theta v$ and $\theta = \alpha$ μ_α-a. e. Since any K^2_θ-function can be uniquely recovered from its boundary values in $L^2(\mu_\alpha)$, (6) holds.

5) Follows from the formula

$$K\sigma_\alpha = \frac{1}{1-\bar{\alpha}\theta^2} = \frac{1}{2}\left(\frac{1}{1-\sqrt{\bar{\alpha}}\theta} + \frac{1}{1-(-\sqrt{\bar{\alpha}})\theta}\right) = \frac{1}{2}\left(K\mu_{\sqrt{\alpha}} + K\mu_{-\sqrt{\alpha}}\right) \blacktriangle$$

To conclude this section let us mention that families $\{\sigma_\alpha\}_{\alpha\in\mathbb{T}}$ have many interesting properties and applications in complex analysis, functional analysis and perturbation theory. Such families were first introduced by D. Clark in [C] and further studied by D. Sarason [S], A. Aleksandrov [A1-5] and the author [P1-6].

3. Examples.

Example 1. Our first example will show that the statement of Theorem 3 fails for $p > 2$. As we will see, the existence of "bad" measures immediately follows from Theorem 1.

If $\sigma \in M_+(\mathbb{T})$ is a non discrete singular measure then for any $p > 2$ there exists $f \in L^p(\sigma)$ such that $C_\sigma f \notin H^p$. Put $\mu = \sigma + m$. Let us define function F:

$$F(\xi) = \begin{cases} f(\xi), & \text{for } \sigma\text{-a. e. } \xi \in \mathbb{T} \\ 0, & \text{for } m\text{-a. e. } \xi \in \mathbb{T} \end{cases}$$

Then $F \in L^p(\mu)$. But since $|K\sigma| \geq \frac{1}{2}||\sigma||$ in $\mathbb{D}$

$$|C_\mu F| = \left|\frac{Kf\mu}{K\mu}\right| = \left|\frac{Kf\sigma}{K\sigma+1}\right| \geq \frac{\frac{1}{2}||\sigma||}{\frac{1}{2}||\sigma||+1}|C_\sigma f|.$$

Since $C_\sigma f \notin H^p$, $C_\mu F \notin L^p(\mu)$.

Example 2. Our next example will demonstrate the unboundedness of maximal transform MC_μ for $p > 2$. We will construct $\mu \in M_+(\mathbb{T})$ and $f \in L^\infty(\mu)$ such that $MC_\mu f \notin L^p(\mu)$ for any $p > 2$.

Consider a sequence of points $\{a_n\}$ in $\mathbb{D}$ satisfying

$$a_n = r_n e^{i\phi_n}, \quad \phi_n = \frac{1}{n}, \quad r_n = 1 - \frac{1}{n^3\ln^2(n+1)}.$$

Then

$$(8) \qquad \sum_n \frac{1-|a_n|^2}{|1-a_n|^2} < \sum_n \frac{\frac{1}{n^3\ln^2(n+1)}}{1/n^2} = \sum_n \frac{1}{n\ln^2(n+1)} < \infty.$$

Let B be the Blaschke product with zeros at $\{a_n\}$ and at 0. As was shown in [A2], (8) implies that B has a non tangential limit of absolute value 1 and non tangential derivative at point 1. Let $B(1) = \alpha$. As we discussed in the previous section, the existence of non tangential derivative at this point implies that the measure $\sigma_\alpha \in M_B$ has a point mass at 1.

Put $\mu = \sigma_\alpha + m$ and define $f \in L^\infty(\mu)$ as 1 at point 1 and as 0 elsewhere. Then

$$(9) \qquad |C_\mu f(a_n)| = \left|\frac{\mu(\{1\})\frac{1}{1-a_n}}{K\mu(a_n)}\right| = \left|\frac{\mu(\{1\})\frac{1}{1-a_n}}{\frac{1}{1-\bar{\alpha}\theta}+1}\right| = \left|\frac{\mu(\{1\})\frac{1}{1-a_n}}{2}\right| \geq Cn$$

for some $C > 0$. For each n consider an arc I_n centered at $\frac{a_n}{|a_n|}$ of the length $\frac{1}{2n^3\ln^2(n+1)}$. By (9) we have that for any $\xi \in I_n$

$$(10) \qquad MC_\mu f(\xi) \geq |C_\mu f(a_n)| \geq Cn.$$

Let now $p = 2+\epsilon$, $\epsilon > 0$. Then, since I_n are disjoint,

$$||MC_\mu f||^p_{L^p(\mu)} \geq ||MC_\mu f||^p_{L^p(m)} \geq \sum_n |I_n|(Cn)^p = \sum_n \frac{1}{2n^{1-\epsilon}\ln^2(n+1)} = \infty.$$

As one can see, in the above example $MC_\mu f$ may still belong to $L^2(\mu)$. It leaves open the question about strong type (2,2). While the answer is unknown, or if it is negative, it would also be interesting to verify if MC_μ is bounded as an operator from $L^p(\mu)$ to $L^2(\mu)$ for $p > 2$ or at least for $p = \infty$.

Example 3. This simple example illustrates the "singularity" of our problem. By Theorem 5 for any $\mu, \nu \in M_+(\mathbb{T})$ $M\frac{K\nu}{K\mu} \in L^{1,\infty}(\mu)$. Here we will show that there exist measures μ and ν from $M_+(\mathbb{T})$ such that

$$M\frac{K\nu}{P\mu} \notin L^p(\mu)$$

for any $p > 0$.

By standard argument it is enough to show that for any $C > 0$ there exist μ and ν such that

$$(11) \qquad \left|\left|M\frac{K\nu}{P\mu}\right|\right|_{L^p(\mu)} \geq C$$

for any $p > 0$.

Put $\nu = \delta_1$. Consider the arc $I = \{e^{i\phi} | \epsilon < \phi < 1\}$, where the positive number ϵ is chosen so small that

$$\int_I \frac{1}{|1-\xi|} dm(\xi) > C.$$

Let $f(\xi) = \frac{1}{|1-\xi|}$ on I and $f = 0$ elsewhere. Consider probability measure $\mu = \frac{fm}{||f||_1}$. Then for any $\xi \in I$ (i. e. μ-a. e.)

$$M\frac{K\nu}{P\mu}(\xi) \geq \left|\frac{K\nu}{P\mu}(\xi)\right| = ||f||_1 > C.$$

Therefore (11) holds for any $p > 0$.

Example 4. Our next construction complements Theorem 1.

Let $\mu \in M_+(\mathbb{T})$, $||\mu|| \leq 1$ be a singular measure whose closed support has a nontrivial interior, i. e. contains an open arc I. We will prove that then C_μ is unbounded as an operator $L^p(\mu) \mapsto H^p$ for any $p > 2$.

We will show that by the choice of $f \in L^p(\mu)$, $||f||_{L^p(\mu)} \leq 1$ the norm $||C_\mu f||_{H^p}$ can be made arbitrarily big. Let θ be an inner function such that $\mu = \mu_1 \in M_\theta$. Consider $\mu_{-1} \in M_\theta$. There exists a closed set $G \subset I$ such that $\mu(G) = m(G) = 0$ and $\mu_{-1}(I \setminus G) < \epsilon$ where ϵ is a small positive constant to be chosen later (recall that all μ_α are singular and $\mu_\alpha \perp \mu_\beta$ for $\alpha \neq \beta$).

Let us now choose a real function $f \in L^p(\mu)$ such that $|f| = 1$ on I and $f = 0$ elsewhere. By assigning plus and minus signs to the values of f on I we can make the boundary values of $Kf\mu$ very small outside of I as well as on G: we can chose f so that

$$|Kf\mu(\xi)| < \epsilon$$

for any $\xi \in \mathbb{T}$, $\xi \notin I \setminus G$. Since $f \in L^2(\mu)$, we have that $C_\mu f \in H^2$, the boundary values of $C_\mu f$ are in $L^2(\mu_{-1})$ and

$$||C_\mu f||^2_{L^2(\mu_{-1})} = ||C_\mu f||^2_{L^2(\mu_1)} = \mu(I).$$

Hence,

$$\int_{I\setminus G} |C_\mu f|^2 \, d\mu_{-1} \geq \int_{\mathbb{T}} |C_\mu f|^2 \, d\mu_{-1} - \epsilon(1-\epsilon) > \mu(I) - \epsilon.$$

By Hölder's inequality

$$\int_{I\setminus G} |C_\mu f|^2 \, d\mu_{-1} \leq (\mu_{-1}(I \setminus G))^{1-\frac{2}{p}} \left(\int_{I\setminus G} |C_\mu f|^p \, d\mu_{-1}\right)^{\frac{2}{p}} \leq \epsilon^{1-\frac{2}{p}} ||C_\mu f||^2_{L^p(\mu_{-1})}.$$

Therefore,

$$||C_\mu f||_{L^p(\mu_{-1})} \geq \sqrt{\frac{\mu(I) - \epsilon}{\epsilon^{1-\frac{2}{p}}}}. \tag{12}$$

It was shown in [A2] (follows from Theorem 1 by duality argument) that

$$||C_\mu f||_{H^p} \geq D||C_\mu f||_{L^p(\mu_{-1})}$$

for some positive constant D which does not depend on f. The last two inequalities imply that by choosing ϵ small we can make $||C_\mu f||_{H^p}$ as big as we want.

4. Proofs.

We start with the proof of the maximal Theorem 4. We only have to verify weak types (2,2) and (1,1). Then the rest of the statement will follow from the standard interpolation theorem, see for instance [SW].

The proof of weak type (2,2) is readily obtained from a result in [P1].

Proof of Theorem 4: weak type (2,2). Let first μ be a singular measure. Then $\mu = \mu_1 \in M_\theta$ for some inner function θ. As was shown in [P1], for every function f from K^2_θ

$$||Mf||_{L^{2,\infty}} \leq C||f||_{H^2} \tag{13}$$

for some absolute constant C. Since C_μ is a unitary operator from $L^2(\mu)$ onto K^2_θ,

$$||f||_{H^2} = ||f||_{L^2(\mu)}. \tag{14}$$

Together the last 2 formulas give us the statement.

If μ is an arbitrary positive measure we can consider a sequence of singular measures tending to μ in the $*$-weak topology of the space of measures. Since the statement holds for each singular measure in the sequence and since the constant C in (13) is absolute we can pass to the limit. ▲

The proof of weak type (1,1) is more complicated. The reason for that is the absence of Clark's Theorem. For $p < 2$ the operator C_μ is not a surjection onto the corresponding K^p_θ. Thus the proof of weak type (1,1) can not directly use properties of pseudocontinuable functions from $K^{1,\infty}_\theta$ as in the proof of type (2,2) above.

Before we complete the proof of Theorem 4 we need some additional preparation.

Lemma 7 (Harnak's Lemma for Cauchy integrals). *Let $\mu \in M_+(\mathbb{T})$ and B be a closed disk inside $\mathbb{D}$ of hyperbolic radius less than $r_0 < 1$. Denote $M = \max_B |K\mu|$ and $m = \min_B |K\mu|$. Then*

$$\frac{M}{m} < C < \infty$$

for some constant C depending only on r_0.

Proof. WLOG $||\mu|| = 1$. Since μ is positive, there exists $\phi \in H^\infty, ||\phi|| \leq 1, \phi(0) = 0$ such that $\mu = \mu_1 \in M_\phi$. Then

$$2K\mu = \frac{1}{1-\phi} = 1 + \frac{1+\phi}{1-\phi}.$$

By the Schwartz Lemma map $\phi : \mathbb{D} \mapsto \mathbb{D}$ does not increase hyperbolic distances. Also, the map $w \mapsto \frac{1+w}{1-w}$ preserves hyperbolic distances as a map from $\mathbb{D}$ to the right half plane, which implies the statement.▲

In fact, one can replace the Cauchy integral in the statement of Lemma 7 with any ratio of Cauchy integrals of positive measures:

Corollary 8. *Let $\mu, \nu \in M_+(\mathbb{T})$ and B be a closed disk inside $\mathbb{D}$ of hyperbolic radius less than $r_0 < 1$. Denote $F = \frac{K\nu}{K\mu}$. Let $M = \max_B |F|$ and $m = \min_B |F|$. Then*

$$\frac{M}{m} < C < \infty$$

for some constant C depending only on r_0.

Proof. Follows from the fact that

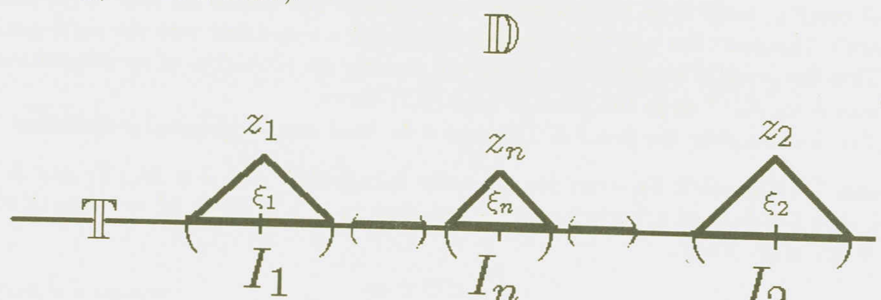
$$M \leq \frac{\max_B |K\nu|}{\min_B |K\mu|}, \quad m \geq \frac{\min_B |K\nu|}{\max_B |K\mu|}$$

and Lemma 7. ▲

Notation. Let g be a function defined in $\mathbb{D}$, $I = \{I_n\}$, $I_1, I_2, \ldots \subset \mathbb{T}$ be a sequence of disjoint open arcs. We denote by g_I the following step function defined on $\mathbb{T}$. Let ξ_n be the center of the arc I_n.

$$g_I(\xi) = \begin{cases} |g((1 - \frac{|I_n|}{2})\xi_n)|, & \text{for } \xi \in I_n \\ 0, & \text{for } \xi \notin \dot{\cup} I_n \end{cases}$$

In other words, the value of g_I on I_n is the value of $|g|$ in the vertex z_n of the triangle with base I_n(see Pic. 2 below).



Pic. 2

One of the obvious properties of g_I is that $g_I \leq Mg$. On the other hand we have

Lemma 9. *Let $F = \frac{K\nu}{K\mu}$ where $\mu, \nu \in M_+(\mathbb{T})$. Suppose that for some $\sigma \in M_+(\mathbb{T})$ $MF \notin L^{1,\infty}(\sigma)$. Then there exists a sequence of disjoint open arcs $I = \{I_n\}$, $I_1, I_2, \ldots \subset \mathbb{T}$ such that $F_I \notin L^{1,\infty}(\sigma)$.*

Proof. By standard argument it is enough to show that F_I can have arbitrarily large $L^{1,\infty}(\sigma)$ norm.

Let $C > 0$. There exists $t > 0$ such that $\sigma(\{MF > t\}) > C/t$. By Corollary 8 for each $\xi \in \{MF > t\}$ there exists a point $r\xi, 0 < r < 1$ on the corresponding radius such that $|F(r\xi)| \geq t/C_1$ for some absolute constant C_1. Denote by I_ξ the open arc of length

r centered at ξ. Consider a cover of $\{MF > t\}$ with such arcs. By Young's Lemma from such a cover one can choose a sequence of *disjoint* arcs $I = \{I_n\}$ such that

$$\sigma(\{MF > t\} \cap \cup I_n) \geq 1/2\sigma(\{MF > t\}).$$

It is left to notice that the $L^{1,\infty}(\sigma)$-norm of F_I is $\geq fracC2C_1$. ▲

Notation. Let f be a function in $\mathbb{D}$ and Σ be a subset of $\mathbb{D}$. We denote by $M_\Sigma f$ the restricted maximal function:

$$M_\Sigma f(\xi) = \sup_{z \in \Gamma_\xi \cap \Sigma} |f(z)|.$$

Now we can use Lemma 9 to obtain:

Lemma 10. *Let $\mu \in M_+(\mathbb{T}), f \in H^{1,\infty}$. Then the following conditions are equivalent:*
1) $Mf \in L^{1,\infty}(\mu)$;
2) for any $C > 0$ $M_{\{|P\mu| > C\}} f \in L^{1,\infty}(\mu)$;
3) there exists $C > 0$ such that $M_{\{|P\mu| > C\}} f \in L^{1,\infty}(\mu)$.

Proof. The implications 1)⇒2)⇒3) are trivial. Let us prove 3)⇒1). Suppose that 3) holds, but $Mf \notin L^{1,\infty}(\mu)$. Then by Lemma 9 $f_I \notin L^{1,\infty}(\mu)$ for some collection of disjoint open arcs $I = \{I_n\}$. As before, z_n will stand for the vertices of triangles, based on I_n, see Pic. 2.

WLOG we can assume that $P\mu(z_n) > C$ for all n. Indeed, if for some n we have $P\mu(z_n) < C$ then, by properties of Poisson kernel, $\mu(I_n) < 3C|I_n|$. But since $(f)_I \leq Mf \in L^{1,\infty}(m)$, we have that the restriction of f_I on the union of all arcs I_n such that $P\mu(z_n) < C$ has finite $L^{1,\infty}(\mu)$-norm. Therefore if we exclude this set from our consideration and consider $J = \{I_n : P\mu(z_n) > C\}$ the function f_J will still have infinite $L^{1,\infty}(\mu)$-norm. Hence we can assume that for every n $P\mu(z_n) > C$.

It is left to notice, that if $P\mu(z_n) > C$ for all n, then $M_{\{|P\mu| > C\}} f \geq f_I \notin L^{1,\infty}(\mu)$. ▲

Lemma 11. *Let $\mu, \nu \in M(\mathbb{T})$, $\mu > 0$. Then $M\frac{P\nu}{P\mu} \in L^{1,\infty}(\mu)$.*

Proof. WLOG ν is positive. Consider Hardy-Littlewood maximal function:

$$\mathcal{M}_{\nu,\mu}(\xi) = \sup \frac{\nu(I)}{\mu(I)},$$

where sup is taken over all arcs I containing ξ. By the standard argument $M\frac{P\nu}{P\mu} \leq \mathcal{M}_{\mu,\nu}$. Hence it is enough to show that $\mathcal{M}_{\nu,\mu} \in L^{1,\infty}(\mu)$.

WLOG μ is a probability measure. Let first μ be a continuous measure. Consider a 1-1 map F from $\mathbb{T}$ onto itself defined as $F(e^{i\phi}) = \mu(\{e^{i\psi} : 0 < \psi < \phi\}$. Map F sends ν into a measure $\eta = F(\nu)$ on $\mathbb{T}$ (as usual, $F(\nu)$ is defined as $[F(\nu)](I) = \nu(F^{-1}(I)))$ and sends μ into Lebesgue measure m. The statement $\mathcal{M}_{\mu,\nu} \in L^{1,\infty}(\mu)$ is now equivalent to the standard result $\mathcal{M}_{\eta,m} \in L^{1,\infty}(m)$, which is true for any measure η with the estimate of the norm depending only on $||\eta|| = ||\nu||$.

The case of arbitrary μ can be proved by constructing a sequence of continuous measures tending to μ in the $*$-weak topology and passing to the limit. ▲

Now we are ready to finish the proof.

Proof of Theorem 4: weak type (1,1).

Suppose that $MC_\mu f \notin L^{1,\infty}(\mu)$ for some $f \in L^1(\mu)$.

WLOG μ is a probability measure. Let again θ be an inner function such that $K\mu = \frac{1}{1-\theta}$. In other words, we consider inner θ such that $\mu = \mu_1$ for the corresponding family of measures $\{\mu_\alpha\}_{\alpha\in\mathbb{T}}$. Since $||\mu|| = 1$, $\theta(0) = 0$.

WLOG f is positive. Choose $g \in L^2(\mu)$ so that $g^2 = f$ and $\int_{\mathbb{T}} g d\mu = 0$. Then $C_\mu g$ is a hermitian element in K^2_θ. Consider $(C_\mu g)^2$. Since $C_\mu g \in K^2_\theta$, by Lemma 6 $(C_\mu g)^2 \in K^1_{\theta^2}$ and $(C_\mu g)^2 = u + \theta v$ for some functions $u, v \in K^{1,\infty}_\theta$. Also by Lemma 6 $C_\mu g^2 = C_\mu f = u + v$. By the part of Theorem 4 that we already proved, $MC_\mu g \in L^{2,\infty}(\mu)$ and therefore $M(C_\mu g)^2 \in L^{1,\infty}(\mu)$. Now we will show that the difference $C_\mu f - (C_\mu g)^2$ is "small" and obtain a contradiction with our assumption that $MC_\mu f \notin L^{1,\infty}(\mu)$.

Let $\lambda \in \mathbb{D}$ and consider the reproducing kernel k_λ of the space K_θ:

$$k_\lambda(z) = \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \bar{\lambda} z}.$$

Let $\{\sigma_\alpha\}$ be the family of measures associated with θ^2. Note that then by Lemma 6

$$\sigma_\alpha = 1/2(\mu_{\sqrt{\alpha}} + \mu_{-\sqrt{\alpha}}). \tag{15}$$

We have $(C_\mu g)^2 \in K^1_{\theta^2}$ and $k_\lambda \in K^\infty_\theta \subset K^\infty_{\theta^2}$. Since the boundary values of $C_\mu g$ exist μ_α-a. e. and belong to $L^2(\mu_\alpha)$ for any $\alpha \in \mathbb{T}$, the boundary values of $(C_\mu g)^2 = u + \theta v$ exist σ-a. e. and belong to $L^1(\sigma_\alpha)$ for any $\alpha \in \mathbb{T}$. Therefore

$$u(z) = \int_{\mathbb{T}} (u + \theta v)\bar{k}_z dm =$$

$$\int_{\mathbb{T}} (u + \theta v)\bar{k}_z d\sigma_1 = \int_{\mathbb{T}} (u + \theta v)\bar{k}_z d\sigma_{-1}. \tag{16}$$

Since $C_\mu g$ is a hermitian element in K^2_θ, (5) and (15) imply that the function $\theta(C_\mu g)^2 = \theta(u + \theta v)$ is real σ_1-a. e. and imaginary σ_{-1}-a. e. Dividing the last equation by $\theta(z)$, using the formula for k_z and taking imaginary parts on both sides we obtain:

$$\operatorname{Im}\left(\frac{1}{\theta}K((u+\theta v)\sigma_1)\right) - Q\left(\bar{\theta}(u+\theta v)\sigma_1)\right) = \operatorname{Im}\left(\frac{1}{\theta}K((u+\theta v)\sigma_{-1})\right) - P\left(\bar{\theta}(u+\theta v)\sigma_{-1})\right).$$

Regrouping and multiplying by $1 - \theta$ we get

$$(1-\theta)Q\left(\bar{\theta}(u+\theta v)\sigma_1)\right) = (1-\theta)P\left(\bar{\theta}(u+\theta v)\sigma_{-1})\right) +$$

$$+(1-\theta)\operatorname{Im}\left(\frac{1}{\theta}K((u+\theta v)\sigma_1)\right) - \operatorname{Im}\left(\frac{1}{\theta}K((u+\theta v)\sigma_{-1})\right). \tag{17}$$

Denote $\Sigma = \{z \in \mathbb{D} | P\mu(z) > 2\}$. Now we will use (17) to show that

$$M_\Sigma\left((1-\theta)Q\left(\bar{\theta}(u+\theta v)\sigma_1)\right)\right) \in L^{1,\infty}(\mu).$$

Let us start with the second summand in the righthand side. By the definition of θ we have that $|\theta| > 1/3$ on Σ. Hence

$$(II) = M_\Sigma\left((1-\theta)\operatorname{Im}\left(\frac{1}{\theta}K((u+\theta v)\sigma_1)\right)\right) = M_\Sigma\left((1-\theta)\frac{u+\theta v}{(1-\theta)\theta}\right) =$$

$$= M_\Sigma\left(\frac{u+\theta v}{\theta}\right) \leq 3M(u+\theta v) \in L^{1,\infty}(\mu).$$

Similarly, since $\left|\frac{1-\theta}{1+\theta}\right| < 1/2$ on Σ, for the third summand we obtain:—

$$(III) = M_\Sigma\left((1-\theta)K(\frac{1}{\theta}(u+\theta v)\sigma_{-1})\right) = M_\Sigma\left((1-\theta)\frac{u+\theta v}{\theta(1+\theta)}\right) \leq$$

$$\leq \frac{3}{2}M(u+\theta v) \in L^{1,\infty}(\mu).$$

Finally for the first summand we have

$$|(1-\theta)P\left(\bar{\theta}(u+\theta v)\sigma_{-1})\right)| \leq 2\frac{|1-\theta|^2}{1-|\theta|^2}|P\left(\bar{\theta}(u+\theta v)\sigma_{-1})\right| = 2\left|\frac{P\left(\bar{\theta}(u+\theta v)\sigma_{-1}\right)}{P\sigma_1}\right|. \tag{18}$$

Using Lemma 11 we obtain

$$(I) = M_\Sigma\left((1-\theta)P\left(\bar{\theta}(u+\theta v)\sigma_{-1})\right)\right) \leq 2M_\Sigma\frac{P\left(\bar{\theta}(u+\theta v)\sigma_{-1}\right)}{P\sigma_1} \in L^{1,\infty}(\sigma_1). \tag{19}$$

By (15) that implies that $(I) \in L^{1,\infty}(\mu)$. Hence

$$M_\Sigma\left((1-\theta)Q\left(\bar{\theta}(u+\theta v)\sigma_1)\right)\right) \leq (I) + (II) + (III) \in L^{1,\infty}(\mu).$$

Now let us recall that

$$(1-\theta)u = (1-\theta)\int_{\mathbb{T}}(u+\theta v)\bar{k}_z d\sigma_1 = (1-\theta)\frac{1}{\theta}K((u+\theta v)\sigma_1) - (1-\theta)K\left(\bar{\theta}(u+\theta v)\sigma_1)\right) =$$

$$= (1-\theta)\frac{1}{\theta}K((u+\theta v)\sigma_1) - (1-\theta)\left(\frac{1}{2}Q\left(\bar{\theta}(u+\theta v)\sigma_1)\right) + \frac{1}{2}P\left(\bar{\theta}(u+\theta v)\sigma_1)\right) + \frac{1}{2}\right).$$

Hence

$$M_\Sigma\left((1-\theta)u\right) \le (II) + \frac{1}{2}M_\Sigma\left((1-\theta)Q\left(\bar\theta(u+\theta v)\sigma_1\right)\right) +$$

$$+\frac{1}{2}M_\Sigma(1-\theta) + \frac{1}{2}M_\Sigma\left((1-\theta)P\left(\bar\theta(u+\theta v)\sigma_1\right)\right).$$

The first two summands on the righthand side were already shown to belong to $L^{1,\infty}(\mu)$. The third one is a maximal function of a bounded function. Using the same ideas as in (18) and (19), the last summand can be estimated as follows:

$$M_\Sigma\left((1-\theta)P\left(\bar\theta(u+\theta v)\sigma_1\right)\right) \le M_\Sigma\left(\frac{P\left(\bar\theta(u+\theta v)\sigma_1\right)}{P\sigma_1}\right) \in L^{1,\infty}(\mu).$$

Therefore

$$M_\Sigma\left((1-\theta)u\right) \in L^{1,\infty}(\mu). \tag{20}$$

Since $M(u-\theta v) \in L^{1,\infty}(\mu)$, we have

$$M\left((1-\theta)(u+\theta v)\right) \in L^{1,\infty}(\mu).$$

Since

$$M_\Sigma\left((1+\theta)\theta v\right) \le M_\Sigma\left((1-\theta)(u+\theta v)\right) + M_\Sigma\left((1-\theta)u\right)$$

we obtain that $M_\Sigma\left((1-\theta)\theta v\right) \in L^{1,\infty}(\mu)$. Since $|\theta| > 1/3$ on Σ,

$$M_\Sigma\left((1-\theta)\theta v\right) \ge 1/3 M_\Sigma\left((1-\theta)v\right) \in L^{1,\infty}(\mu).$$

Since

$$M_\Sigma\left(C_\mu f\right) \le M_\Sigma\left((C_\mu g)^2\right) + M_\Sigma\left(C_\mu f - (C_\mu g)^2\right) = M_\Sigma\left((C_\mu g)^2\right) + M_\Sigma\left((1-\theta)v\right)$$

and

$$M_\Sigma\left((C_\mu g)^2\right) \le M_\Sigma\left((C_\mu g)^2\right) \in L^{1,\infty}(\mu),$$

we have that $M_\Sigma\left(C_\mu f\right) \in L^{1,\infty}(\mu)$. Therefore, by Lemma 10 $M\left(C_\mu f\right) \in L^{1,\infty}(\mu)$ and we obtain a contradiction. ■

Remark. By Lemma 10, (20) implies that $M\left((1-\theta)u\right) \in L^{1,\infty}(\mu)$. This is the centeral point of the proof. Recalling that $K\mu = \frac{1}{1-\theta}$ it translates to:

$$\frac{u}{K\mu} \in L^{1,\infty}(\mu). \tag{21}$$

One may ask a more general question: for what functions u does (21) hold?

One can easily show that (21) holds for all $u \in H^1$. Indeed, if u is summable on $\mathbb{T}$ then it is representable by Poisson integral and we have

$$\frac{u}{K\mu} = \frac{Pu}{K\mu} \le \frac{Pu}{P\mu} \in L^{1,\infty}(\mu)$$

by Lemma 11. In our last proof, however, u is not from H^1 but from $H^{1,\infty}$. Unfortunately, one eventually realizes that there exist functions from $H^{1,\infty}$ not satisfying (21). Even after we recall that u belongs to a smaller space $K_\theta^{1,\infty}$ and ask if (21) holds for all such u, the answer is still negative. The fact that makes our proof work is that u belongs to even smaller set, namely, to the projection of $\left(K_\theta^2\right)^2$ on $K_\theta^{1,\infty}$:

$$u \in \{\theta P_-\bar\theta f^2 : f \in K_\theta^2\}$$

where P_- is the Riesz projection. We have proved that (21) holds for all u from this set.

Theorem 5 shows that (21) also holds for all u representable by Cauchy integrals.

Proof of Theorem 5. Since $|K\mu| \ge \frac{1}{2}||\mu||$ in $\mathbb{D}$ the statement is equivalent to

$$M\frac{K\nu}{K(\mu+m)} \in L^{1,\infty}(\mu). \tag{22}$$

By Theorem 4 the operator $C_{\mu+m}$ has weak type (1,1). It is left to consider a sequence of positive functions $f_n \in L^1(\mu+m)$ such that $f_n(\mu+m) \to \nu$ in $*$-weak topology of the space of measures and pass to the limit. ■

Our last remaining proof follows from Theorem 4 and a result from [P].

Proof of Theorem 3.

The case $p=2$, once again, can be handled with the methods from [P1] because one has an opportunity to use Clark's Theorem. In fact, a more general result is proved in [P1]: if $f \in L^2(\mu)$ then the Fourier series $\sum_{n\ge 0} a_n z^n$ of the function $C_\mu f$ converge in $L^2(\mu)$. The limit of the Fourier series obviously coincides with the boundary values of $C_\mu f$ almost everywhere with respect to the absolutely continuous part of μ. In addition, by Theorem 2, the boundary values of $C_\mu f$ are equal to f almost everywhere with respect to the singular part of μ. Hence $C_\mu f \in L^2(\mu)$

Weak type (1,1) follows from Theorem 4 and the rest follows from the interpolation theorem. ■

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