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SPECTRAL MEASURES AND CATEGORY

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Abstract

We analyze spectral properties of self–adjoint operators under rank one perturbations. We show that for any self–adjoint operator with dense spectrum and any standard family of its rank one perturbations there exists a first category set supporting the spectral measures of all the operators in the family. An analogous result is true for the spectra of a family of Sturm–Liouville operators corresponding to various boundary conditions.

Introduction

This note analyzes spectral behavior of a selfadjoint operator under a rank one perturbation and shows how the same methods can be applied to study the perturbations of spectra of ordinary differential expressions when the boundary condition at one end changes.

Consider a selfadjoint operator A in a separable Hilbert space, a vector φ and the corresponding family of selfadjoint rank one perturbations of A:

$$A + \lambda(\varphi, \cdot)\varphi, \ \lambda \in \mathbf{R}.$$

In many applications of perturbation theory, operator A satisfies an additional condition that its spectrum contains an interval $I \subset \mathbf{R}$. It is well known that then there exists a first category set (see [10] for precise definitions), which contains all the embedded eigenvalues on I of all the operators A_λ, $\lambda \in \mathbf{R}$, see [2], [7]. This fact was used in particular, to prove existence of singular continuous spectrum for large sets of boundary conditions for ordinary differential expressions and to answer some questions about the perturbational behavior of the essential spectrum.

In regard of this result, it is natural to ask whether the other parts of the spectra have similar properties. In this paper we give a complete answer to this question. Namely, we prove that there exists a first category set which supports all spectral measures of all the operators A_λ, $\lambda \in \mathbf{R}$ on I. Our result implies the before mentioned property of embedded eigenvalues as well as some other known results on the perturbations of spectra. We shall give a general argument which includes the cases of rank one perturbations and Sturm–Liouville operators. The unified approach developed in [9], [5] which needs the negative part of the potential to be infinitesimally form bounded with respect to $-\frac{d^2}{dx^2}$, will not be used. We will need only elementary calculations.

Our results could be considered as a contribution to the understanding of the rich interplay between the concepts of measure and category in spectral theory. We proceed as follows. In Section 1 we introduce the families of measures that we are going to study. These diagonalize the corresponding operators and appear in the Poisson integral representation of harmonic functions related to the resolvent operator. In Section 2 we prove our main results about the existence of a common support of first category. In Section 3 we give explicit example where such support can be constructed. This support will be the set of points where there are subordinate solutions, a concept generalizing the idea of eigenfunctions.

Section 1

Let ϕ be a nonconstant analytic function in the upper half plane $\mathbf{C}_+$ such that $|\phi| \leq 1$. Then for any α with $|\alpha| = 1$ the function

$$i\frac{\alpha+\phi}{\alpha-\phi}$$

has positive imaginary part in $\mathbf{C}_+$.

Thus, according to a well known representation theorem, see for example [13] page 83, there exists a measure μ_α such that

$$Im\left(i\frac{\alpha+\phi(z)}{\alpha-\phi(z)}\right) = c\,y + \frac{1}{\pi}\int_{\mathbf{R}} \frac{y}{(t-x)^2+y^2} d\mu_\alpha(t) \tag{1}$$

and

$$\int_{\mathbf{R}} \frac{d\mu_\alpha(t)}{1+t^2} < \infty \quad , \quad c \geq 0 \quad z = x+iy.$$

In what follows we shall study two cases where families of measures of the above type appear as spectral measures of self–adjoint operators.

Case a) Let $\mathcal{H}$ be a Hilbert space, $\mathcal{H}_0$ a self–adjoint operator on $\mathcal{H}$ which may or may not be bounded and P a self–adjoint projection on a fixed normalized element k. For all $x \in \mathcal{H}$ we have $Px = (x,k)k$. Consider the family of operators $H_\lambda = H_0 + \lambda P$ where λ is real.

Let us introduce the resolvent operator

$$R_z^\lambda = (H_\lambda - zI)^{-1}$$

and the functions

$$F_\lambda(z) = (R_z^\lambda k, k) = \int_{\mathbf{R}} \frac{d\nu_\lambda(t)}{t-z}$$

which are analytic in the upper and lower half–planes and have a positive imaginary part in the upper half–plane. The measure ν_λ is the spectral measure corresponding to k given by the spectral theorem.

The following relation holds. See [4], [15] (1.13)

$$F_\lambda(z) = \frac{F_0(z)}{1+\lambda F_0(z)}.$$

We can write

$$F_0(z) = i\frac{1+\phi(z)}{1-\phi(z)}$$

where

$$\phi(z) = \frac{F_0(z)-i}{F_0(z)+i}.$$

The function ϕ is analytic and satisfies $|\phi(z)| \leq 1$.

A simple calculation gives the following equality

$$(1+\lambda^2)\frac{i\frac{1+\phi}{1-\phi}}{1+i\lambda\frac{1+\phi}{1-\phi}} - \lambda \quad = \quad i\frac{\frac{1+i\lambda}{1-i\lambda}+\phi}{\frac{1+i\lambda}{1-i\lambda}-\phi}. \tag{2}$$

Notice that $\left|\frac{1+i\lambda}{1-i\lambda}\right| = 1$.

The relation between the spectral measures ν_λ and the measures μ_α which appeared in the integral representation (1) is then given by

$$(1+\lambda^2)d\nu_\lambda = d\mu_\alpha$$

where

$$\alpha = \frac{1+i\lambda}{1-i\lambda}.$$

Case b) Let us consider now operators generated by the differential expression

$$lu = -u'' + q(x)u \qquad 0 \leq x < \infty$$

(where q is real valued, locally integrable function defined in $[0,\infty)$) and the boundary conditions at zero

$$u(0)\cos\theta + u'(0)\sin\theta = 0 \qquad 0 \le \theta < \pi.$$

We assume that the limit point case occurs at infinity.

For each θ there is a function $m_\theta(z)$ which is analytic and with positive imaginary part in the upper half plane, called the Weyl–m function.

The imaginary part of $m_\theta(z)$ has the following integral representation

$$Im\ m_\theta(z) = \int_{\mathbf{R}} \frac{\mu}{(t-\xi)^2+\mu^2} d\rho_\theta(t) \qquad z = \xi + i\mu.$$

The measures ρ_θ are the Weyl spectral measures of the boundary value problem. These measures diagonalize the corresponding operators and contain all the information about the spectrum. The following relation is satisfied:

$$m_\theta(z) = \frac{-\sin(\theta-\beta) + m_\beta(z)\cos(\theta-\beta)}{\cos(\theta-\beta) + m_\beta(z)\sin(\theta-\beta)}. \tag{3}$$

See [1], [15].

After some elementary calculations this equality can be rewritten, for $\beta = 0$, as

$$i\frac{e^{2i\theta}+\psi(z)}{e^{2i\theta}-\psi(z)} = m_\theta(z)$$

where $\psi(z) = \frac{m_0(z)-i}{m_0(z)+i}$. Observe that $|\psi(z)| \le 1$.

Therefore $\rho_\theta = \mu_\alpha$ where $\alpha = e^{2i\theta}$ and μ_α are the measures in the expression (1).

Section 2

Using equality (2) we obtain

$$Im\left(i\frac{\alpha+\phi}{\alpha-\phi}\right) \le \frac{1+\lambda^2}{\lambda^2}\cdot\frac{1}{\varepsilon}$$

if

$$Im\left(i\frac{1+\phi}{1-\phi}\right) > \varepsilon \quad , \quad \lambda \ne 0$$

where

$$\alpha = \frac{1+i\lambda}{1-i\lambda}$$

and from (1) we get then,

$$cy + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(t-x)^2+y^2} d\mu_\alpha(t) \le \frac{1+\lambda^2}{\lambda^2}\cdot\frac{1}{\varepsilon}. \tag{4}$$

If we define for $z \in \mathbf{C}, z = x + iy$ the set

$$V_z = \left\{ p \in \mathbf{R} : |x - p| < \frac{1}{2} y \right\}$$

we can obtain the inequalities

$$\begin{aligned} cy \quad + & \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(t-x)^2 + y^2} d\mu_\alpha(t) \geq \\ & \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(t-x)^2 + y^2} d\mu_\alpha(t) \geq \\ & \frac{1}{\pi} \int_{V_z} \frac{|V_z|}{(t-x)^2 + |V_z|^2} d\mu_\alpha \geq \\ & \frac{4}{5\pi} \frac{\mu_\alpha(V_z)}{|V_z|} \quad z = x + iy. \end{aligned}$$

Therefore together with inequality (4) we get

$$\frac{4}{5\pi} \frac{\mu_\alpha(V_z)}{|V_z|} \leq \frac{1 + \lambda^2}{\lambda^2} \cdot \frac{1}{\varepsilon}$$

that is

$$\mu_\alpha(V_z) \leq |V_z| \left(\frac{5\pi}{4} \frac{1 + \lambda^2}{\lambda^2} \right) \cdot \frac{1}{\varepsilon} \tag{5}$$

if $Im\ i\frac{1+\phi}{1-\phi} > \varepsilon \quad$ for $\alpha \neq 1$.

We say that S is a support of a measure μ or that μ is supported on S if

$$\mu(\mathbf{R} \backslash S) = 0.$$

A set S is a closed support if it is closed and it is a support.

Lemma 1 Assume that μ is a $\sigma -$ finite measure on $\mathbf{R}$ and that S is a closed support of μ without isolated points. Then there exists $\tilde{S} \subset S$ such that $\tilde{S}$ is of first category in S and $\mu(\mathbf{R} \backslash \tilde{S}) = 0$.

Proof Let $P = \{p \in S / \mu(p) > 0\}$ Since μ is $\sigma -$ finite , P is at most countable. We are assuming that S is perfect (closed and without isolated points), therefore the set $S \backslash P$ is dense in S. Since any subset of $\mathbf{R}$ is separable, we can choose a countable dense subset $\{a_1, a_2, \ldots\}$ of $S \backslash P$. This set is dense in S.

For each $n, m \in \mathbf{N}$ we choose an open interval I_{nm} with center a_n, such that

$$\mu(I_{nm}) < \frac{1}{2^{n+m}}.$$

This can be done because $\mu(a_n) = 0$ and, for any nested sequence of intervals I_i,

$$\mu(I_i) \to \mu(\bigcap_{j=1}^{\infty} I_j) \qquad \text{as } i \to \infty$$

holds.

Let

$$G_m = \bigcup_{n=1}^{\infty} (I_{nm} \cap S).$$

Then

$$\mu(G_m) \leq \sum_{i=1}^{\infty} \mu(I_{im}) \leq \frac{1}{2^m} \overset{m\to\infty}{\longrightarrow} 0\,.$$

Therefore,

$$\mu(\bigcap_{m=1}^{\infty} G_m) = 0$$

and the lemma follows with $\tilde{S} = \mathbf{R} \setminus \bigcap_{m=1}^{\infty} G_m$. □

Inequality (5) now allows to prove the following result which shows the existence of a "small" support for all measures μ_α.

Theorem 1 *Assume that $I \subset \mathbf{R}$ is an interval such that for every subinterval $J \subset I$ we have $\mu_{\alpha_0}(J) > 0$, for some $\alpha_0 \in \partial D = \{z \in \mathbf{C} : |z| = 1\}$.*

Then there exists a set $S \subset I$, S of first category such that

$$\mu_\alpha(I \setminus S) = 0$$

holds for every $\alpha \in \partial D$.

Proof Without loose of generality we can choose $\alpha_0 = 1$.

First we will show the existence of a countable set $\{\lambda_n\}_{n=1}^{\infty}$, dense in I such that

$$Im\, i\frac{1+\phi(\lambda_n + i0)}{1-\phi(\lambda_n + i0)} > \varepsilon_n > 0$$

where

$$\phi(\lambda + i0) := \lim_{y\to 0_+} \phi(\lambda + iy).$$

Recall that (see [11] p. 44)

$$Im\, i\frac{1+\phi(\lambda + i0)}{1-\phi(\lambda + i0)} = \pi \lim_{\varepsilon\to 0_+} \frac{\mu(\lambda-\varepsilon, \lambda+\varepsilon]}{2\varepsilon}$$

(where $\mu := \mu_{\alpha=1}$).

From a theorem of De la Vallée Poussin we know that

$$\mu\left(\left\{\lambda / \lim_{\varepsilon\to 0_+} \frac{\mu(\lambda-\varepsilon,\lambda+\varepsilon]}{2\varepsilon} = 0\right\}\right) = 0, \tag{6}$$

See [14] p. 127.

If $\mu(J) > 0$ then according to (6), there exists $\lambda_0 \in J$ such that

$$Im\ i\frac{1+\phi(\lambda_0+i0)}{1-\phi(\lambda_0+i0)} > \varepsilon_0 > 0.$$

Since this happens for every subinterval $J \subset I$ we can construct the set $\{\lambda_n\}_{n=1}^{\infty}$ mentioned above.

Now for $n, m \in \mathbf{N}$ define

$$z_n^m = \lambda_n + i\ y_n^m \in \mathbf{C}$$

where we choose y_n^m such that

$$y_n^m < \varepsilon_n \frac{1}{2^{n+m}}$$

and

$$Im\ i\frac{1+\phi(z_n^m)}{1-\phi(z_n^m)} > \varepsilon_n > 0.$$

Let us define

$$G_m = \bigcup_{n=1}^{\infty} (V_{z_n^m} \cap I)$$

the set G_m is open and dense in I.

We have then

$$\mu_\alpha(G_m) \le \mu_\alpha\left(\bigcup_{n=1}^{\infty} V_{z_n^m}\right) \le \sum_{n=1}^{\infty} \mu_\alpha(V_{z_n^m}).$$

Using inequality (5) we get

$$\begin{aligned}\sum_{n=1}^{\infty}\mu_\alpha(V_{z_n^m}) &\le \frac{5\pi}{4}\frac{1+\lambda^2}{\lambda^2}\sum_{n=1}^{\infty}\frac{|V_{z_n^m}|}{\varepsilon_n}\\ &< \frac{5\pi}{4}\frac{1+\lambda^2}{\lambda^2}\sum_{n=1}^{\infty}\frac{1}{2^{n+m}} = \frac{5\pi}{4}\frac{1+\lambda^2}{\lambda^2}\frac{1}{2^m} \overset{m\to\infty}{\to} 0\end{aligned}$$

therefore $\mu_\alpha(G_m) \overset{m\to\infty}{\to} 0$ and $\mu_\alpha\left(\bigcap_{m=1}^{\infty} G_m\right) = 0$ for every $\alpha \neq 1$.

Hence μ_α is supported in the set $I \setminus \bigcap_{m=1}^{\infty} G_m$ which is of first category *i.e.* countable union of nowhere dense sets (see [10]).

Since any single $\sigma-$finite measure may be supported on a first category set, according to Lemma 1, we have the desired support for $\mu_\alpha, \alpha = 1$. Taking the union of this set and the common support for all other $\alpha' s$, the result follows. □

From here it follows in particular that all the eigenvalues of rank one perturbations live in a first category set.

Corollary Let A be a selfadjoint operator whose closed spectrum contains an interval $I \subset \mathbf{R}$. Let φ be cyclic vector for A and $A_\lambda = A + \lambda(\varphi, \cdot)\varphi, \lambda \in \mathbf{R}$, be the corresponding rank one perturbations of A. Then there exits a first category subset F of I such that for any operator A_λ the part of its spectral measure ν_λ in I, is supported on F.

Proof Straightforward from Theorem 1 and the relation

$$(1 + \lambda^2) d\nu_\lambda = d\mu_\alpha$$

mentioned at the end of case a) in Section 1.

□

Remark A simple example of a family of measures *not* supported on a first category set is given by $\{\mu_p\}_{p \in [0,1]}$ where

$$\mu_p(A) = \begin{cases} 1 & \text{if } \; p \in A \\ 0 & \text{if } \; p \notin A \end{cases} .$$

In some cases, a description of the first category set which supports the measures μ_θ can be given, as we will see in the next section.

Section 3

The next theorem shows that, when we have only singular spectrum in an interval I, the set of points corresponding to subordinate solutions is a first category support for all the measures μ_θ. This is not true if the spectrum has empty interior. See remark at the end of this section. We follow the notation introduced in Section 1 case b).

We recall the following (see [6]):

Definition A solution $u_s(r, z)$ of $lu = zu$ is said to be subordinate at infinity if, for every linearly independent solution $u(r, z)$,

$$\lim_{N \to \infty} \frac{||u_s(r, z)||_N}{||u(r, z)||_N} = 0$$

where $|| \; f(r) \; ||_N$ denotes $(\int_0^N |f(r)|^2 dr)^{1/2}$.

We shall need the following:

Lemma 2 Assume $S \subset \mathbf{R}$. If there exists $\varepsilon > 0$ such that

$$S_\varepsilon = \{x \in S \mid Im\ m_\beta(x + i0) > \varepsilon\}$$

is dense in S, then

$$B = \{x \in S \mid Im\ m_\beta(x + i0) = 0\}$$

is of first category in S.

Proof Let

$$A_{nk} = \left\{x \in S \mid Im\ m_\beta\left(x + i\frac{1}{k}\right) \leq \frac{1}{n}\right\}.$$

Then we have, for every $N \in \mathbf{N}$,

$$B \subset \bigcap_{n=N}^{\infty}\left[\bigcup_{j=1}^{\infty}\left(\bigcap_{k\geq j} A_{nk}\right)\right]. \tag{7}$$

We shall prove that the set on the right is of first category in S.

Let $N \in \mathbf{N}$ be such that $N > \frac{1}{\varepsilon}$.

Then, for any $J \in \mathbf{N}$,

$$S_\varepsilon \subset \bigcup_{k\geq J} A^c_{nk} \quad \text{if } n > N$$

where we take the complement of A_{nk} with respect to S, i.e.,

$$A^c_{nk} = \left\{x \in S \mid Im\ m_\beta\left(x + i\frac{1}{k}\right) > \frac{1}{n}\right\}.$$

Since S_ε is dense in S, we know that $\bigcap\limits_{n\geq J} A_{nk}$ has empty interior with respect to S. Since, moreover, $\bigcap\limits_{k\geq J} A_{nk}$ is closed with respect to S, then it is nowhere dense with respect to S and therefore the set on the right of (7) is of first category with respect to S and so is B. □

Let $\sigma(L_\beta)$ and $\sigma_{ac}(L_\beta)$ denote the spectrum and the absolutely continuous spectrum of L_β respectively. See [8] for precise definitions. Recall that μ_β is purely singular if and only if σ_{ac} is empty.

Theorem 2 Let $I \subset \mathbf{R}$ be an interval. Assume $I \subset \sigma(L_\beta)$ and $\sigma_{ac}(L_\beta) \cap I = \emptyset$.

Let

$$S = \{x \in I \mid \text{ a subordinate solution of } lu = xu \text{ exists }\}.$$

Then S is of first category with respect to I and

$$\mu_\theta(I \backslash S) = 0 \qquad \forall\theta.$$

Proof Let us consider the sets

$$B = \{x \in I \mid Im\ m_\beta(x + i0) = 0\}$$

$$C = \{x \in I \mid |m_\beta(x + i0)| = \infty\}.$$

According to Theorem 1 Section 6 of [6] we have that $S = B \cup C$. From Theorem 3(iii) of [6] it follows that

$$\mu_\theta(I \backslash S) = 0 \qquad \forall \theta.$$

Therefore, it is enough to prove that $B \cup C$ is of first category with respect to I.

To prove that B is of first category, recall that the set

$$\{x \in I \mid Im\ m_\beta(x + i0) = \infty\}$$

is dense in I, since it is a support of the singular part of the spectral measure (see Proposition 1(iii) of [6]).

Applying now Lemma 2 it follows that B is of first category in I. From formula (3) we know that

$$C \subset \{x \in I \mid Im\ m_\theta(x + i0) = 0\}$$

for $\theta \neq \beta$.

Since $\sigma_{ac}(L_\beta) \cap I = \emptyset$, we know that L_θ has only singular spectrum in I. It then follows that

$$\{x \in I \mid Im\ m_\theta(x + i0) = \infty\}$$

is dense in I.

Using Lemma 2 again, we obtain that

$$\{x \in I \mid Im\ m_\theta(x + i0) = 0\}$$

is of first category in I and, therefore, the set C is of first category with respect to I. □

Open Problem: Do we really need assumption $\sigma_{ac}(L_\beta) \cap I = \emptyset$ in Theorem 2?

It is hard to believe that existence of absolutely continuous spectrum can turn the set S into second category.

Remark If $\sigma(L_\beta)$ has empty interior, then it is possible to have for every point of an interval I a subordinate solution, even if $\sigma_{ess}(L_\beta) \cap I$ is "big", (where $\sigma_{ess}(L_\beta)$ denotes the set of accumulation points of $\sigma(L_\beta)$). See [3], [12].

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