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Stability of a Singular Continuous Spectrum of Sturm–Liouville Operators

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Abstract. We prove that there exists a continuous potential q such that the operator generated by

$$(lu)(x) = -u''(x) + \{q(x) + v(x)\}u(x), \quad 0 \leqslant x < \infty$$

and boundary conditions $u(0)\cos\alpha + u'(0)\sin\alpha = 0$ has a singular continuous spectrum in $[0, 1]$ for every locally integrable function v with compact support and every $\alpha \in [0, 2\pi)$.

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1. Introduction

In recent years, increasing interest in disordered systems and almost periodic potentials has led to a vigorous literature on the 'pathological' spectrum of Schrödinger operators, such as the singular continuous and dense point spectra. For the deterministic case, the singular continuous spectrum seems to be very unstable under a broad class of perturbations (see, for example, [3]). Nevertheless, the singular continuous spectrum of a Sturm–Liouville operator on the half axis can be stable under an arbitrary local perturbation to the potential, as we shall prove in this Letter.

In Carey and Pincus [2], it is proven that if an operator H has only a singular continuous spectrum, then there is an operator V of arbitrarily small trace norm with $H + V$ pure point. We shall show that an analogous result does not hold for local perturbations.

In Pearson [7], an example of a potential giving rise to a singular continuous spectrum is constructed. Locally, the potential may be arbitrary, but it is not clear if it will keep the singular continuous spectrum for every local perturbation.

2. Statement of the Result

Let $\rho: \mathbf{R} \to \mathbf{R}$ be defined as

$$\rho(\lambda) = \begin{cases} 0, & \text{if } \lambda \in (-\infty, 0], \\ F(\lambda), & \text{if } \lambda \in (0, 1], \\ 1 - \dfrac{2}{\pi} + \dfrac{2\sqrt{\lambda}}{\pi}, & \text{if } \lambda \in (1, \infty), \end{cases}$$

where $F(\cdot)$ is the function defined in Riesz and Nagy [8], p. 49. With $t = 1/3$, the function is singular continuous in $[0, 1]$, where t is a parameter which appears in the construction given in [8].

For convenience, we recall the construction of F.

First, a sequence of functions $F_n: [0, 1] \to \mathbf{R}$ is defined by recursion as follows. Let $F_0(x) = x$. Let $F_n(x)$ be continuous and linear in the segments bounded by two consecutive points $k2^{-n}$, $(k+1)2^{-n}$. Let $F_{n+1}(x) = F_n(x)$ at these points, whereas at the midpoint of these segments, that is, at the new points of division, let

$$F_{n+1}\left(\frac{\alpha+\beta}{2}\right) = \tfrac{1}{3}F_n(\alpha) + \tfrac{2}{3}F_n(\beta),$$

where α and β denote the extremities of the respective segment.

The sequence $\{F_n(x)\}$ converges to a limit $F(x)$. The function F is continuous, strictly increasing, and $F'(x) = 0$ almost everywhere (see [8]). The function ρ satisfies the conditions of the theorem of Gelfand–Levitan (see [6]). Therefore there exists a continuous potential $q: \mathbf{R}^+ \to \mathbf{R}$, where $\mathbf{R}^+$ denotes the positive reals, and $\alpha \in [0, 2\pi)$ such that the operator L generated by the differential expression

$$(lu)(x) = -u''(x) + q(x)u(x), \quad 0 \leqslant x < \infty,$$

and the boundary condition

$$u(0) \cos \alpha + u'(0) \sin \alpha = 0$$

has ρ for its spectral function.

Let $v: \mathbf{R}^+ \to \mathbf{R}$ be a locally integrable function with compact support $S \subset \mathbf{R}^+$.

Let us define the self-adjoint operator $\tilde{L}$ as the one generated by the differential expression

$$\tilde{l}u = -u'' + \{q(x) + v(x)\}u, \quad x \in [0, \infty),$$

and the boundary condition

$$u(0) \cos \beta + u'(0) \sin \beta = 0, \quad \beta \in [0, 2\pi).$$

Let $I := [0, 1]$. Our main result is the following theorem.

THEOREM. *For every $\beta \in [0, \pi)$ and every locally integrable function v of compact support, the operator $\tilde{L}$ has a singular continuous spectrum in I.*

3. Proof of the Theorem

For the proof of the theorem, we shall need some lemmas.

LEMMA 1. *Let ρ be defined as above. Then*

$$\int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{(\lambda - \xi)^2} = \infty$$

for every $\xi \in [0, 1]$.

Proof. Let ξ be an arbitrary point in [0, 1] and consider the sequence of nested intervals $I_n = (\alpha_n, \beta_n)$ of the type

$$\alpha_n = k2^{-n}, \qquad \beta_n = (k+1)2^{-n},$$

about x. It happens that

$$\frac{F(\beta_n) - F(\alpha_n)}{\beta_n - \alpha_n} = \prod_{k=1}^{n} (1 + \varepsilon_k \tfrac{1}{3}),$$

where $\varepsilon_k = \pm 1$.

Now, since $\beta_n - \alpha_n = (\frac{1}{2})^n$, we have

$$\frac{F(\beta_n) - F(\alpha_n)}{(\beta_n - \alpha_n)^2} = 2^n \prod_{k=1}^{n} (1 + \varepsilon_k \tfrac{1}{3}) \geqslant (\tfrac{4}{3})^n \to \infty, \quad n \to \infty.$$

Now,

$$\int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{(\lambda - \xi)^2} \geqslant \int_{I_n} \frac{d\rho(\lambda)}{(\lambda - \xi)^2} \geqslant \frac{1}{|I_n|^2} \int_{I_n} d\rho(\lambda) = \frac{\rho(I_n)}{|I_n|^2} = \frac{F(\beta_n) - F(\alpha_n)}{(\beta_n - \alpha_n)^2},$$

where $|I_n|$ denotes the length of the interval I_n and we have used the letter ρ to denote the function and the measure generated by the function.

Taking the limit when n goes to infinity yields the assertion of the lemma. □

The essential spectrum of an operator T, denoted by $\sigma_{\text{ess}}(T)$, is the set of accumulation points of $\sigma(T)$. The absolutely continuous spectrum of T, denoted by $\sigma_{\text{ac}}(T)$, is defined as the spectrum of T restricted to its subspace of absolute continuity (see [5]).

The following result holds.

LEMMA 2. $\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}(\tilde{L})$ *and* $\sigma_{\text{ac}}(L) = \sigma_{\text{ac}}(\tilde{L})$.

For the proof see for example [4].

LEMMA 3. *If λ is an eigenvalue of $\tilde{L}$, then λ is an eigenvalue of L for some boundary condition at zero.*

The easy proof is left to the reader.

Proof of the Theorem. If $\alpha = \beta$ and $v \equiv 0$, then $L = \tilde{L}$ and the theorem holds because of the way we constructed the spectral function of L.

Since the spectrum of L is purely singular in I from Lemma 2, it follows that $\tilde{L}$ must have only a singular spectrum in I. We have then only to prove that this spectrum is continuous.

Assume now that $\tilde{L}$ has an eigenvalue $\lambda \in I$. From Lemma 3, it follows that λ is an eigenvalue of L for some boundary condition. But Lemma 1, together with Theorem 4 of Aronszajn [1], imply that L cannot have eigenvalues in I. Therefore, the spectrum of $\tilde{L}$ is purely singular continuous. □

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