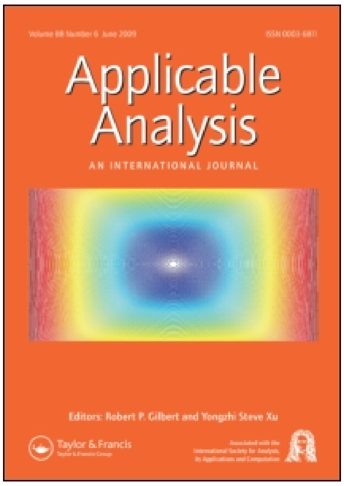

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Sturm–Liouville operators in the half axis with shifted potentials

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We consider Sturm–Liouville operators in the half axis generated by shifts of the potential and prove that Lebesgue measure is equivalent to a measure defined as an average of spectral measures which correspond to these operators. This is then used to obtain results on stability of spectral types under change of parameters such as boundary conditions, local perturbations, and shifts. In particular if for a set of shifts of positive measure the corresponding operators have α-singular, singular continuous and (or) point spectrum in a fixed interval, then this set of shifts has to be unbounded. Moreover, there are large sets of boundary conditions and local perturbations for which the corresponding operators enjoy the same property.

Keywords: Sturm–Liouville operator; Spectral measure; Singular spectrum; Shifted potentials

2000 Mathematics Subject Classifications: 34L40; 34B05; 34B24; 34L05; 47E05

1. Introduction

In several proofs of localization phenomena a key step has been to establish absolute continuity of measures $\mu(\cdot) = \int \rho_\lambda(\cdot)\mathrm{d}\lambda$ generated as averages of spectral measures ρ_λ which correspond to selfadjoint operators, particularly to Schrödinger operators. Using various versions of an argument known as Kotani's trick it can be shown that the continuous spectrum is absent in some models. Depending on the case, the averaging parameter λ could be a boundary condition, a coupling constant in a family $H_0 + \lambda W$ or number related to a shift of the potential. See [6,7,9,10,12,13].

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Consider for example a family of self-adjoint operators $L_\theta(q)$ in $L_2(0,\infty)$ generated by the differential expression

$$lu = -u'' + q(x)u$$

and the boundary condition

$$u(0)\cos\theta - u'(0)\sin\theta = 0, \quad \theta \in [0,\pi),$$

and let ρ_θ be the spectral measure corresponding to L_θ. The average $\mu(\cdot) := \int_0^\pi \rho_\theta(\cdot)\mathrm{d}\theta$ is not only absolutely continuous with respect to Lebesgue measure, but is equal to it! (See e.g. [5,11, Theorem 1]).

Using the absolute continuity of μ the following can be proven (see e.g. [2, Corollary 3.2]).

Theorem 1 *Let I be an open set in $\mathbb{R}$. If for almost every $E \in I$, there exists a nontrivial L^2 solution of $lu = Eu$, then*

$$\sigma_c(L_\theta) \cap I = \phi$$

for almost every $\theta \in [0,\pi)$, where σ_c denotes the continuous spectrum.

After realizing the usefulness of the absolute continuity of μ it is natural to ask if Lebesgue measure is absolutely continuous with respect to μ and if this could be of any use to understand spectral properties of the operators involved.

In this article, we give conditions which imply that Lebesgue measure and a properly chosen average measure μ have the same zero sets and apply this to prove stability results concerning spectra, when potentials of half line Schrödinger operators are shifted. In particular, if some spectral property holds for a set of positive Lebesgue measure of shifts then the same property will hold for a set of positive Lebesgue measure of other parameters, such as boundary conditions and local perturbations.

This will be a consequence of our main result Theorem 2. See for example Corollary 1 below.

In previous work [3] a similar approach was used to study local perturbations with compact support. Here we shall consider situations where both shift and local perturbations are present. In section 2 our main result is proven. The techniques rely heavily on the Prüfer transform which is an important tool in this article. In section 3, we show how more general results can be obtained and some applications to α-singular and α-continuous spectrum. Some of the methods used have their origin in [10] and [13].

We shall use the notation $\mathbb{R}^+$ for the non-negative reals, i.e. $\mathbb{R}^+ = \{a \in \mathbb{R}/a \geq 0\}$.

2. Main results

For each $(a,\theta) \in \mathbb{R}^+ \times [0,\pi)$ let us consider the self-adjoint operator $L_{a\theta}$ in $L_2(0,\infty)$ generated by the differential expression

$$l_a = \frac{-\mathrm{d}^2}{\mathrm{d}x^2} + V_a(x),$$

where

$$V_a(x) := \begin{cases} q(x-a) & \text{if } x \geq a \\ 0 & 0 \leq x < a \end{cases}$$

as follows:

$L_{a\theta}u = l_a u$ for u in the domain of $L_{a\theta}$ given by

$$D(L_{a\theta}) = \big\{ f \in L^2(0,\infty) : f, f' \text{ locally absolutely continuous in } (0,\infty), \\ l_a f \in L_2(0,\infty),\ f(0)\cos\theta - f'(0)\sin\theta = 0,\ \theta \in [0,\pi) \big\}.$$

We assume $lu = -u'' + q(x)u$ is in the limit point case at ∞. The function $q(x)$ is local in L_1.

If μ and ν are two measures we use the notation $\mu \prec\prec \nu$ when μ is absolutely continuous with respect to ν, that is if $\nu(A) = 0$ implies $\mu(A) = 0$. If $\mu \prec\prec \nu$ and $\nu \prec\prec \mu$ we say that the two measures are equivalent. We use $|\cdot|$, to denote the Lebesgue measure.

Let $a_2 > a_1 > 0$ and $E_0 > 0$. For a Borel set $A \subset [E_0, \infty)$ define

$$\mu_\theta(A) := \int_{a_1}^{a_2} \rho_{a\theta}(A)\mathrm{d}a,$$

where $\rho_{a\theta}$ is the spectral function associated with the operator $L_{a\theta}$.

THEOREM 2

(i) $\mu_\theta(\cdot) \prec\prec |\cdot|$
(ii) *If* $a_2 - a_1 \geq (\pi/\sqrt{E_0})$ *then* $\mu_\theta(\cdot) \succ\succ |\cdot|$.

Before we prove the theorem let us introduce notation and recall some results.

Consider real solutions $u \not\equiv 0$ of $l_a u = Eu$ such that for fixed $c \in \mathbb{R}^+$

$$u(c) = \sin\theta$$
$$u'(c) = \cos\theta.$$

If we write the vector $(u'(x), u(x))$ for $x > 0$ in polar coordinates we obtain

$$u(x) = r_c(x)\sin\phi_c(x)$$
$$u'(x) = r_c(x)\cos\phi_c(x)$$

where $\phi_c(x,\theta,E,V_a)$ and $r_c(x,\theta,E,V_a)$ are called the Prüfer phase and the Prüfer amplitude of u, respectively.

We fix a unique value of ϕ_c by requiring $\phi_c(c,\theta,E) = \theta$ and continuity in x. These functions r_c and ϕ_c are jointly continuous in x, E (use arguments similar to [15, Thm 2.1], [1, Thm Ch. 2.4], [4, Thm Ch. V.3]). This will be important in what follows.

We shall need the next results

(a)

$$\rho_{a\theta}((E_1, E_2)) = \lim_{b\to\infty} \frac{1}{\pi} \int_{E_1}^{E_2} r_0(b, \theta, E, V_a)^{-2}\, \mathrm{d}E$$

if E_1 and E_2 are not discrete points of $\rho_{a\theta}$. See [11, Thm 2], $E_1 < E_2$.

(b) For any $a, x, \theta, E \in \mathbb{R}$

$$\frac{1}{\pi} \int_{\theta}^{\theta+\pi} r_a(x, \beta, E, V_a)^{-2}\, \mathrm{d}\beta = 1.$$

See [13, Corollary 12] and [14, Appendix B].

(c)

$$\frac{\partial \phi_0}{\partial x} = \cos^2 \phi_0(x) + (E - V_a) \sin^2 \phi_0(x).$$

This follows from a straightforward calculation.

Proof of Theorem 2
We use an argument similar to the one used in [13] where (I) was proven for the case of the whole line. Given any $E_1 < E_2, 0 < E_0 \le E_1$ the measure $\rho_{a\theta}$ is continuous in E_1 and E_2 for almost any a.

From (a) above we know that

$$\mu(E_1, E_2) = \int_{a_1}^{a_2} \rho_{a\theta}(E_1, E_2)\mathrm{d}a = \int_{a_1}^{a_2} \left[\lim_{b\to\infty} \frac{1}{\pi} \int_{E_1}^{E_2} r_0(b + a, \theta, E, V_a)^{-2}\, \mathrm{d}E \right] \mathrm{d}a \qquad (1)$$

We shall forget for a while the limit which appears in the expression (1). Our aim is to get first the estimate (5) below.

Since

$$\begin{aligned} r_0\big(b + a, \theta, E, V_a\big) &= r_0\big(a, \theta, E, 0\big) r_a\big(b + a, \phi_0(a, \theta, E, 0), E, V_a\big) \\ &= r_0\big(a, \theta, E, 0\big) r_0\big(b, \phi_0(a, \theta, E, 0), E, q(x)\big) \end{aligned} \qquad (2)$$

we obtain

$$\begin{aligned} &\int_{a_1}^{a_2} \frac{1}{\pi} \left[\int_{E_1}^{E_2} \big(r_0(b + a, \theta, E, V_a)\big)^{-2}\, \mathrm{d}E \right] \mathrm{d}a \\ &\quad = \int_{a_1}^{a_2} \mathrm{d}a \frac{1}{\pi} \left[\int_{E_1}^{E_2} \mathrm{d}E \Big(r_0\big(a, \theta, E, 0\big)^{-2} r_0\big(b, \phi_0(a, \theta, E, 0), E, q(x)\big)^{-2} \Big) \right] \end{aligned}$$

Since $r_0(a,\theta,E,0)$ is uniformly bounded for $(a,E)\in[a_1,a_2]\times[E_1,E_2]$ (use joint continuity) and interchanging the order of the integrals, we get that

$$\begin{aligned} &C_1\int_{E_1}^{E_2}\mathrm{d}E\int_{a_1}^{a_2}\mathrm{d}a\, r_0\big(b,\phi_0(a,\theta,E,0),E,q(x)\big)^{-2}\\ &\geq \int_{a_1}^{a_2}\frac{1}{\pi}\left[\int_{E_1}^{E_2}\big(r_0(b+a,\theta,E,V_a)\big)^{-2}\,\mathrm{d}E\right]\mathrm{d}a\\ &\geq C_2\int_{E_1}^{E_2}\mathrm{d}E\int_{a_1}^{a_2}\mathrm{d}a\, r_0\big(b,\phi_0(a,\theta,E,0),E,q(x)\big)^{-2} \end{aligned} \tag{3}$$

where we can choose, for example, C_1 and C_2 to be the sup, respectively the inf, of $r_0(a,\theta,E,0)$ when $(a,E)\in[a_1,a_2]\times[E_1,E_2]$.

Now if we denote

$$\beta(a):=\phi_0(a,\theta,E,0)$$

and change variables we obtain

$$\int_{a_1}^{a_2}\mathrm{d}a\, r_0\big(b,\beta(a),E,q(x)\big)^{-2}=\int_{\beta(a_1)}^{\beta(a_2)}\frac{\mathrm{d}\beta}{\beta'(a)}r_0\big(b,\beta(a),E,q(x)\big)^{-2}.$$

Since

$$\min\{1,E\}\leq\beta'(a)\leq\max\{1,E\},$$

(see (c) aforementioned) and recalling that $E\in[E_1,E_2]$ then

$$\begin{aligned} C_3\int_{\beta(a_1)}^{\beta(a_2)} r_0\big(b,\beta,E,q(x)\big)^{-2}\,\mathrm{d}\beta &\geq \int_{a_1}^{a_2}\mathrm{d}a\, r_0\big(b,\beta(a),E,q(x)\big)^{-2}\\ &\geq C_4\int_{\beta(a_1)}^{\beta(a_2)} r_0\big(b,\beta,E,q(x)\big)^{-2}\,\mathrm{d}\beta \end{aligned} \tag{4}$$

for suitable positive constants C_3, C_4.

From inequalities (3) and (4) we obtain

$$\begin{aligned} C_5\int_{E_1}^{E_2}\mathrm{d}E\left[\int_{\beta(a_1)}^{\beta(a_2)} r_0\big(b,\beta,E,q(x)\big)^{-2}\mathrm{d}\beta\right] &\geq \int_{a_1}^{a_2}\frac{1}{\pi}\left[\int_{E_1}^{E_2}\big(r_0(b+a,\theta,E,V_a)\big)^{-2}\,\mathrm{d}E\right]\mathrm{d}a\\ &\geq C_6\int_{E_1}^{E_2}\mathrm{d}E\left[\int_{\beta(a_1)}^{\beta(a_2)} r_0\big(b,\beta,E,q(x)\big)^{-2}\,\mathrm{d}\beta\right] \end{aligned} \tag{5}$$

for some positive constants C_5, C_6.

Proof of (I)

Splitting the interval $[\beta(a_1), \beta(a_2)]$ into intervals of length at most π yields by (b) aforementioned

$$\int_{\beta(a_1)}^{\beta(a_2)} r_0\big(b, \beta, E, q(x)\big)^{-2}\, \mathrm{d}\beta \le \pi\left(\frac{\beta(a_2) - \beta(a_1)}{\pi} + 1\right),$$

therefore from the first inequality in (5) we get

$$C_7(E_2 - E_1) \ge \int_{a_1}^{a_2} \mathrm{d}a\left[\frac{1}{\pi}\int_{E_1}^{E_2} r_0(b + a, \theta, E, V_a)^{-2}\, \mathrm{d}E\right].$$

Applying Fatou's lemma we obtain

$$C_7(E_2 - E_1) \ge \mu_\theta(E_1, E_2).$$

Using countable additivity part (I) of the theorem follows for general Borel sets.

Proof of (II)

Let us look now for conditions on a_1, a_2 which imply $\beta(a_2) - \beta(a_1) \ge \pi$. Since in the definition of β the potential zero was used, it is possible to calculate β explicitly.

Observe that $u(x) := \sin(\alpha + \sqrt{E}x)$, where $\alpha := \arctan(\sqrt{E}\tan\theta) \in [0, \pi)$, satisfies the equation

$$-u''(x) = Eu(x)$$

and the conditions

$$u(0) = \sin\theta$$
$$u'(0) = \cos\theta.$$

We can then write

$$\begin{aligned}\beta(x) + n\pi &= \phi_0(x, \theta, E, 0) + n\pi \\ &= \arg\big(u'(x) + iu(x)\big) + n\pi \\ &= \arg\big(\sqrt{E}\cos(\alpha + \sqrt{E}x) + i\sin(\alpha + \sqrt{E}x)\big) + n\pi \\ &= \arg\big(\sqrt{E}\cos(\alpha + \sqrt{E}x + n\pi) + i\sin(\alpha + \sqrt{E}x + n\pi)\big) \\ &= \beta\left(x + \frac{n\pi}{\sqrt{E}}\right).\end{aligned}$$

Since $\beta(x)$ is increasing (see (c) aforementioned) then $a_2 - a_1 \ge (\pi/\sqrt{E})$ implies $\beta(a_2) - \beta(a_1) \ge \pi$.

Therefore using (b) we get

$$\int_{\beta(a_1)}^{\beta(a_2)} r_0\big(b, \beta, E, q(x)\big)^{-2}\,\mathrm{d}\beta \geq \pi \quad \text{if } a_2 - a_1 \geq \frac{\pi}{\sqrt{E}}.$$

We can conclude that

$$\int_{a_1}^{a_2} \frac{1}{\pi}\left[\int_{E_1}^{E_2} r_0(b+a, \theta, E, V_a)^{-2}\,\mathrm{d}E\right]\mathrm{d}a \geq C_8(E_2 - E_1) \quad \text{if } a_2 - a_1 \geq \frac{\pi}{\sqrt{E_0}}.$$

To be able to consider the limit which appears in (1) we shall bound

$$\tilde{F}_b(a) := \int_{E_1}^{E_2} r_0(b+a, \theta, E, V_a)^{-2}\,\mathrm{d}E$$

for every $b > 0$ and apply the Lebesgue-dominated convergence theorem. Observe that using the decomposition (2) it is enough to bound

$$F_b(a) = \int_{E_1}^{E_2} r_0\big(b, \beta(a), E, q(x)\big)^{-2}\,\mathrm{d}E.$$

From Lemma 1 of [11] we know that

$$F_b(a) = \int_0^{\pi} \mu^b_{\beta(a)\gamma}(E_1, E_2)\mathrm{d}\gamma$$

where $\mu^b_{\beta\gamma}$ is the spectral measure associated with the regular problem in $[0, b]$ with boundary conditions

$$u(0)\cos\beta - u'(0)\sin\beta = 0$$
$$u(b)\cos\gamma - u'(b)\sin\gamma = 0.$$

In the same article [11] it is observed (see proof of Corollary 3) that these measures are uniformly bounded in b, β, γ. Hence the boundedness of F follows. Therefore

$$\mu_\theta(E_1, E_2) \geq C_8(E_2 - E_1)$$

and using countable additivity we get (II) for general Borel sets. ■

Corollary 1 *Let $I := (E_1, E_2) \subset \mathbb{R}^+$ open and define $L_{a\theta}$ as above. For any $a \in \mathbb{R}^+$, the operator $L_{a\theta}$ has singular continuous spectrum in I for a set of positive Lebesgue measure of θ's, if and only if for any $\theta \in [0, \pi)$, $L_{a\theta}$ has singular continuous spectrum in I for a set B of a's of positive Lebesgue measure.*

$$\textit{Moreover } \big|B \cap [a_1, a_2]\big| > 0 \quad \textit{if } a_2 - a_1 \geq \frac{\pi}{\sqrt{E_0}} \textit{ where } 0 < E_0 < E_1.$$

Proof $\Rightarrow$) Let S be the set of points E for which there are subordinate solutions of $l_a u = Eu$ which are not in L_2. It is known that this set is a support of the singular continuous part of $L_{a\theta}$ and it does not depend on a and θ. Since $\rho_{a\theta}(S \cap I) > 0$ for a set of positive measure in θ by hypothesis, using equality

$$|S \cap I| = \int_0^\pi \rho_{a\theta}(S \cap I)\mathrm{d}\theta$$

we deduce $|S \cap I| > 0$. This implies using Theorem 2 (II) that $\mu_\theta(S \cap I) > 0$ for any θ if $a_2 - a_1 \geq (\pi/\sqrt{E_0})$. From here we know $\rho_{a\theta}(S \cap I) > 0$ for $a \in B$ where B satisfies

$$\left|B \cap [a_1, a_2]\right| > 0$$

$\Leftarrow$) Assume $L_{a\theta}$ has singular continuous spectrum in I for a set B of a's of positive Lebesgue measure. Then $\rho_{a\theta}(S \cap I) > 0$ for $a \in B$ and

$$\int_B \rho_{a\theta}(S \cap I)\mathrm{d}a > 0.$$

Therefore there exists an interval $J = [a_1, a_2]$ such that

$$\int_J \rho_{a\theta}(S \cap I)\mathrm{d}a \geq \int_{B \cap J} \rho_{a\theta}(S \cap I)\mathrm{d}a > 0.$$

Using Theorem 2 (I) we obtain $|S \cap I| > 0$ and therefore

$$\int_0^\pi \rho_{a\theta}(S \cap I)\mathrm{d}\theta = |S \cap I| > 0$$

for every fixed a. Therefore $L_{a\theta}$ has singular continuous spectrum in I for a set of positive Lebesgue measure in θ. ■

Remark If instead of taking the support S as above we take the set P which corresponds to subordinate solutions which are in L_2 we get the same result for the pure point part and taking $P \cup S$, we obtain the result for the singular part of $L_{a\theta}$.

3. Generalizations

Recall the construction of $L_{a\theta}$ given in section 2 and choose $q(x) = V(x) + \lambda W(x)$ where we assume $W(x) > 0$ for a.e. $x \in [0, c]$ and $W(x) = 0$ for every $x \notin [0, c]$. The corresponding operator and spectral function will be denoted by $L_{a\theta\lambda}$ and $\rho_{a\lambda\theta}$ respectively. Fixing two of the three parameters a, λ, θ we define lines in $\mathbb{R}^+ \times \mathbb{R} \times [0, \pi)$ as follows

$$l = \left\{(a, \lambda, \theta) \in \mathbb{R}^+ \times \mathbb{R} \times [0, \pi)/\text{two parameters are fixed. In case } (a, \theta) \text{ are fixed, set } a = 0\right\}.$$

The set of all lines defined in this way will be denoted by $\mathcal{M}$. Now consider the set $\mathcal{P}, \mathcal{P} \subset \mathcal{M}$ defined in the following way

$$\mathcal{P} = \left\{ l \in \mathcal{M} / \exists\ B \subset l, |B| > 0 \text{ such that } \sigma(L_{a,\lambda,\theta}) \cap I \neq \phi \text{ for } (a, \lambda, \theta) \in B \right\}.$$

Here $|\cdot|$ denotes the one-dimensional Lebesgue measure and $\sigma = \sigma_{sc}, \sigma_{pp}$ or σ_s where, as usual, $\sigma_{sc}, \sigma_{pp}, \sigma_s$ denote the singular continuous, pure point, and singular spectrum respectively.

THEOREM 3

$$\mathcal{P} = \mathcal{M} \quad or \quad \mathcal{P} = \phi.$$

Proof If I is open we know that

$$\sigma(L_{a,\lambda,\theta}) \cap I \neq \phi \Longleftrightarrow \rho_{a,\lambda,\theta}(A \cap I) > 0 \tag{6}$$

where $A = S, P$ or $S \cup P$, as defined in Corollary 1 and the Remark following it, depending on whether σ is σ_{sc}, σ_{pp} or σ_s. See for example [2, Corollary 2.8]. On the other hand we know

$$|\cdot| \sim \int_{\lambda_1}^{\lambda_2} \rho_{0\lambda\theta}(\cdot)\mathrm{d}\lambda \sim \int_{a_1}^{a_2} \rho_{a\lambda\theta}(\cdot)\mathrm{d}a \sim \int_0^{\pi} \rho_{a\lambda\theta}(\cdot)\mathrm{d}\theta \tag{7}$$

where $\sim$ denotes equivalence of measures (two measures are equivalent if they have the same sets of measure zero), if we take $\lambda_2 - \lambda_1, a_2 - a_1$ large enough. If not, then the measures defined as averages are just absolutely continuous with respect to Lebesgue measure.

The first equivalence is the main result in [3], the second is Theorem 2 above, and the third goes back to [5].

The theorem then follows from (6) and (7).

Let us for example fix $a = 0, \theta = \theta_0$ and let λ vary in $\mathbb{R}$. Assume there is a set $B \subset \mathbb{R}$ of positive Lebesgue measure such that for $\lambda \in B$

$$\sigma(L_{0,\lambda,\theta_0}) \cap I \neq \phi.$$

In other words, assume that the line $\{(0, \lambda, \theta_0)/\lambda \in \mathbb{R}\}$ is in $\mathcal{P}$. Then we know by (6), that $\rho_{0,\lambda,\theta_0}(A \cap I) > 0$ for $\lambda \in B$. Using that $\int_{\lambda_1}^{\lambda_2} \rho_{0\lambda\theta_0}(\cdot)\mathrm{d}\lambda \prec\prec |\cdot|$ (with no restrictions on the length $\lambda_2 - \lambda_1$) it follows that $|A \cap I| > 0$ and from (7) we obtain

$$\sigma(L_{a,\lambda,\theta}) \cap I \neq \phi$$

for a set of a's of positive measure. The other cases are proven analogously. ∎

In the case $\sigma = \sigma_{pp}$, we can choose

$$q(x) = V(x) + \lambda W(x) + U(x)$$

where U is locally L_1 and

$$\int |U(x)|e^{A|x|}\,\mathrm{d}x < \infty \quad \text{for all } A > 0.$$

Using [8, Corollary 1.8], Theorem 3 can be proven in this case too.

PROPOSITION 1 *Let I be an open interval $I \subset [E_0, \infty)$, $E_0 > 0$ and define*

$$B := \left\{a \in \mathbb{R}^+ / \sigma(L_{a\lambda_0\theta_0}) \cap I \neq \phi\right\}.$$

If $|B| > 0$ then

$$|B \cap [a_1, a_2]| > 0$$

wherever $a_2 - a_1 \geq (\pi/\sqrt{E_0})$. In particular B is unbounded.

Proof From (6), if $a \in B$ then $\rho_{a\lambda_0\theta_0}(A \cap I) > 0$ and $0 < \int_{a_1}^{a_2} \rho_{a\lambda_0\theta_0}(A \cap I)\mathrm{d}a$ for some $a_1 < a_2$ if $|B| > 0$.

From Theorem 2(I) it follows $|A \cap I| > 0$, and taking $a_1 < a_2$ such that $a_2 - a_1 \geq (\pi/\sqrt{E_0})$ then from Theorem 2(II) we get

$$\int_{a_1}^{a_2} \rho_{a\lambda_0\theta_0}(A \cap I)\mathrm{d}a > 0$$

and therefore $|B \cap [a_1, a_2]| > 0$ using (6) again. ∎

Similar results can be obtained for the α-continuous and α-singular spectrum. Recall that for $\alpha \in [0, 1]$ the α-dimensional Hausdorff measure is defined for Borel sets A by

$$h^\alpha(A) \equiv \lim_{\delta \to 0} \inf_{\delta\text{-covers}} \sum_{v=1}^{\infty} |b_v|^\alpha,$$

where a δ-cover is a countable collection of intervals each, at most, of length δ so $A \subset \cup_{v=1}^{\alpha} b_v$.

Given $\alpha \in [0, 1]$ we define a measure μ to be α-continuous (αc) if $\mu(S) = 0$ for any set S with $h^\alpha(S) = 0$ and α-singular (αs) if it is supported on a set of S with $h^\alpha(S) = 0$. For every such α and any measure μ, one can uniquely decompose $\mu = \mu^{\alpha c} + \mu^{\alpha s}$ with $\mu^{\alpha c}\alpha$-continuous and $\mu^{\alpha s}, \alpha$-singular.

Denote $\rho := \rho_{a\lambda\theta}$. It is possible to find sets A_α and B_α such that

$$\mathrm{d}\rho^{\alpha c} = \mathrm{d}\rho(A_\alpha \cap .)$$
$$\mathrm{d}\rho^{sc} = \mathrm{d}\rho(B_\alpha \cap .)$$

and it happens that A_α and B_α are independent of θ, a and λ. See [8] and references therein.

Using the same reasoning as above one can prove Theorem 3 for the α-singular and α-continuous part of the spectral measure. We get for example

THEOREM 4 *If* $\rho^{\alpha c}_{a\lambda_0\theta_0}(I) > 0$ *for* $a \in B$ *where* I *is an open interval,* $|B| > 0$ *and* λ_0, θ_0 *are fixed then* $\rho^{\alpha c}_{a_0\lambda_0\theta}(I) > 0$ *for* $\theta \in \tilde{B}, |\tilde{B}| > 0$ *for any* a_0, λ_0 *fixed.*

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