IMRN International Mathematics Research Notices
1999, No. 11

Corrections and Addendum to "Inverse Spectral Analysis with Partial Information on the Potential, III. Updating Boundary Conditions"

Rafael del Rio, Fritz Gesztesy, and Barry Simon

Unless explicitly stated otherwise, all subsequent formula numbers, references, cited theorems, page numbers, etc., refer to those in [1].

(1) Our proof of Lemma 2.2 contains a number of typographical errors:

(a) $\sinh(2|y|^{1/2})$ should be replaced by $\sinh(|2y|^{1/2})$ in (2.3). Similarly, $\exp(2|y|^{1/2})$ should be replaced by $\exp(|2y|^{1/2})$ in the 2d and 4th lines on p. 755.

(b) The infinite product in (2.4) should read

$$\prod_{n=1}^{\infty}\left[\left(1+\frac{16|y|^2}{\pi^4 n^4}\right)^{1/2}\left(1+\frac{|y|^2}{\lambda_n^2}\right)^{-1/2}\right],$$

and the argument following (2.4) should be replaced by the following one.

If $0 \le a \le b$, then

$$\frac{1+a^2|y|^2}{1+b^2|y|^2} \le 1;$$

and if $a > b > 0$, then

$$\left(\frac{1+a^2|y|^2}{1+b^2|y|^2}\right)^{1/2} = \left(1+\frac{(a^2-b^2)|y|^2}{1+b^2|y|^2}\right)^{1/2} \le \left(1+\frac{a^2-b^2}{b^2}\right)^{1/2} = \frac{a}{b},$$

$$\prod_{n=1}^{\infty}\left(\frac{1+\frac{16|y|^2}{\pi^4 n^4}}{1+\frac{|y|^2}{\lambda_n^2}}\right)^{1/2} \le \prod_{n:\lambda_n>\pi^2 n^2/4}\frac{4\lambda_n}{\pi^2 n^2} = \prod_{n=1}^{\infty}\left(1+\frac{\big(\lambda_n-(\pi^2 n^2/4)\big)_+}{\pi^2 n^2/4}\right) < \infty$$

if (2.1) holds.

Received 23 March 1999.

(c) Replace the estimates (2.5a) and (2.5b) by the asymptotic relations

$$P_j(z) \underset{z\to\pm i\infty}{=} O(\cos(\sqrt{z})) \quad \text{and} \quad Q_j(z) \underset{z\to\pm i\infty}{=} O(\sqrt{z}\sin(\sqrt{z})),$$

respectively.

(d) Replace $f(iy)$ by $|f(iy)|$ in the 4th line of p. 755.

(2) A slight modification of our proof of Theorem 2.1 allows us to also recover Marchenko's uniqueness result (Theorem 2.3.2 in [16]), which goes beyond Borg's theorem (our Corollary 3.2) in the following sense. Marchenko does not a priori fix the boundary conditions when comparing spectra for the two potentials q_1 and q_2. More precisely, denote by $\sigma(q; h, \ell)$ the spectrum of the operator $-d^2/dx^2 + q$ in $L^2((0,1))$ with boundary conditions $u'(0) + hu(0) = 0$, $u'(1) + \ell u(1) = 0$, $h, \ell \in \mathbb{R}$. Then Marchenko's result reads as follows.

Theorem (Theorem 2.3.2 in [16]). Let $q_1, q_2 \in L^1((0,1))$ be real-valued, suppose that $h_j, k_j, \ell_j \in \mathbb{R}$, $j = 1, 2$, $h_1 \neq k_1$, $h_2 \neq k_2$, and assume

$$\sigma(q_1; h_1, \ell_1) = \sigma(q_2; h_2, \ell_2), \qquad \sigma(q_1; k_1, \ell_1) = \sigma(q_2; k_2, \ell_2). \tag{1}$$

Then $h_1 = h_2$, $k_1 = k_2$, $\ell_1 = \ell_2$, and $q_1 = q_2$ a.e. on $[0,1]$. □

Sketch of proof. The asymptotic eigenvalue behavior (3.1) yields

$$h_1 - h_2 = k_1 - k_2 := A.$$

Identifying $m_1(z) = m_{\ell_1}(z)$ and $m_2(z) = m_{\ell_2}(z)$, and redefining $H(z)$ as

$$\widetilde{H}(z) = P_2(z)Q_1(z) - P_1(z)Q_2(z) + AP_1(z)P_2(z)$$

and $G(z)$ as

$$\widetilde{G}(z) = \widetilde{H}(z)/f(z) = P_1(z)P_2(z)[m_1(z) - m_2(z) + A]/f(z),$$

one can follow the proof of Theorem 2.1 step by step. As a result, one arrives at

$$|\widetilde{G}(iy)| \le \frac{\exp(|2y|^{1/2})}{|f(iy)|}|m_1(iy) - m_2(iy) + A| \underset{|y|\to\infty}{=} O(|y|^{-1/2}) \tag{2}$$

and hence at $\widetilde{G}(z) = 0$ using a Phragmén-Lindelöf argument. Thus,

$$m_1(z) - m_2(z) + A = 0,$$

and the asymptotic behavior

$$m_j(z) \underset{z\to\pm i\infty}{=} \pm i\sqrt{z} + o(1)$$

then proves $A = 0$. Hence,

$$m_1(z) = m_2(z) \text{ and } h_1 = h_2, \ k_1 = k_2.$$

But then $m_{\ell_1}(z) = m_{\ell_2}(z)$ yields $\ell_1 = \ell_2$ and $q_1 = q_2$ a.e. on $[0, 1]$ by Marchenko's Theorem 1.1. ■

We emphasize the $O(|y|^{-1/2})$-term in (2), as opposed to the corresponding $o(|y|^{-1/2})$-term in our proof of Theorem 2.1. This shows that in contrast to Borg's theorem (i.e., Corollary 3.2), all eigenvalues are required in (1), and one can no longer dispense with the knowledge of one of them.

Actually, Marchenko also includes the case of Dirichlet boundary conditions by reducing it to the case discussed above by means of appropriate linear fractional transformations of $m(z)$. We omit further details.

Acknowledgments

We thank Professor Alexander Ramm for pertinent questions concerning our paper, which led us to double check the arithmetic in our proof of Lemma 2.2 and also prompted us to derive the present alternative proof of Marchenko's theorem.

References

[1] Rafael del Rio, Fritz Gesztesy, and Barry Simon, *Inverse spectral analysis with partial information on the potential*, *III. Updating boundary conditions*, Internat. Math. Res. Notices (IMRN) **1997**, 751–758.

del Rio: IIMAS-UNAM, Apdo. Postal 20-726, Admon No. 20, 01000 Mexico D.F., Mexico; delrio@servidor.unam.mx

Gesztesy: Department of Mathematics, University of Missouri, Columbia, Missouri 65211, USA; fritz@math.missouri.edu

Simon: Division of Physics, Mathematics, and Astronomy, California Institute of Technology, Pasadena, California 91125, USA; bsimon@caltech.edu