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Inverse Spectral Analysis with Partial Information on the Potential, III. Updating Boundary Conditions

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1 Introduction

This paper is a postscript to two earlier papers [5], [6] in that it provides a new way of looking at the problems considered in those papers; it allows the same methods to prove additional results.

To explain our results, we recall earlier theorems of Borg [1] (see also [8], [10]–[14]) and of Hochstadt-Lieberman [9] (see also [7], [15]). Throughout this paper, assume $q \in L^1((0,1))$ to be real-valued; consider the operator $H = -\frac{d^2}{dx^2} + q$ in $L^2((0,1))$ with boundary conditions

$$u'(0) + h_0 u(0) = 0, \tag{1.1}$$

$$u'(1) + h_1 u(1) = 0, \tag{1.2}$$

where $h_j \in \mathbb{R} \cup \{\infty\}$, $j = 0, 1$ (with $h_0 = \infty$ shorthand for the boundary condition $u(0) = 0$). Fix $h_1 \in \mathbb{R}$ but think of $H(h_0)$ as a family of operators depending on h_0 as a parameter. Then Borg's and Hochstadt-Lieberman's results can be paraphrased as follows.

Borg [1]. The spectra of $H(h_0)$ for two values of h_0 uniquely determine q. $\square$

Hochstadt-Lieberman [9]. The spectra of $H(h_0)$ for one value of h_0 and q on $[0, 1/2]$ determine q. $\square$

In [6], two of us proved a result which can be paraphrased as follows.

Theorem of [6]. Half the spectra of one $H(h_0)$ and q on $[0, 3/4]$ uniquely determine q. $\square$

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One of our goals in this note is to prove the following.

New Result. The spectrum of one $H(h_0)$ and half the spectrum of another $H(h_0)$ and q on $[0, 1/4]$ uniquely determine q.

We will also show the following.

New Result. Two-thirds of the spectra of three $H(h_0)$ uniquely determine q.

Our point is as much a new way of looking at the argument in [6] as these new results. Fundamental to our approach here and in [5], [6] is the Titchmarsh-Weyl m-function defined by

$$m_{h_1}(z) = \frac{u'_{h_1}(z,0)}{u_{h_1}(z,0)},$$

where $u_{h_1}(z,x)$ solves $-u''(z,x) + q(x)u(z,x) = zu(z,x)$ with the boundary condition (1.2). m_{h_1} is a meromorphic function on $\mathbb{C}$ (in fact, a Herglotz function) with all of its zeros and poles on the real axis. Since $h_1 \in \mathbb{R}$ will be fixed throughout this paper, we will delete the subscript h_1 from now on and simply write $m(z)$ instead. Moreover, due to the assumption $h_1 \in \mathbb{R}$, we will index the eigenvalues of $H(h_0)$ by $\{\lambda_n\}_{n\in\mathbb{N}_0}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

A fundamental result of Marchenko [16] (see also [2], [3], [17]) reads as follows.

Theorem 1.1. $m(z)$ uniquely determines q a.e. on $[0, 1]$. □

Our fundamental strategy can be described as follows.

(a) Note that λ is an eigenvalue of $H(h_0)$ if and only if $m(\lambda) = -h_0$.

(b) Prove a general theorem that knowing m at points $\lambda_0, \lambda_1, \dots, \lambda_n, \dots$ determines m as long as $\{\lambda_n\}_{n\in\mathbb{N}_0}$ has sufficient density. Given (a), this will allow one to prove that if $\lambda_0, \lambda_1, \dots, \lambda_n, \dots$ have sufficient density, an infinite sequence of pairs $\{(\lambda_n, \alpha_n)\}_{n\in\mathbb{N}_0}$ and the knowledge that $H(h_0 = \alpha_n)$ has an eigenvalue at λ_n determines m (and so q a.e. on $[0, 1]$ by Theorem 1.1).

(c) Use scaling covariance to extend the $[0, 1]$ result to one for $[x, 1]$ for any $x \in (0, 1)$.

(d) Note that a knowledge of q a.e. on $[0, x]$ allows one to update boundary conditions. Explicitly, let $H(h_x)$ be the operator in $L^2((x, 1))$ with boundary condition (1.2) but (1.1) replaced by

$$u'(x) + h_x u(x) = 0. \tag{1.3}$$

Then λ_n is an eigenvalue of $H(h_0 = \alpha_n)$ if and only if it is an eigenvalue of $H(h_{x_0} = \beta_n)$, where β_n is obtained by solving $m'_n(x) = q(x) - \lambda_n - m_n^2(x)$ on $[0, x_0]$ with the boundary condition $m_n(x = 0) = -\alpha_n$ and setting $\beta_n = -m_n(x = x_0)$.

We will present steps (b) and (c) in Sections 2 and 3 and then step (d) in Section 4.

We will not explicitly derive them, but the results in [6] which treat operators on $(0,1)$ and allow one to trade C^{2k} conditions on q for k eigenvalues, can be extended to the context we discuss here.

We also note that the ideas in this paper extend to Jacobi matrices.

Finally, while the present paper and [5], [6] concentrate on discrete spectra, we might point out that our m-function strategy also applies in certain cases involving absolutely continuous spectra (see [4]).

2 Zeros of the m-function

If $a \in \mathbb{R}$, let $a_+ = \max(a, 0)$.

Theorem 2.1. Let $\{\lambda_n\}_{n\in\mathbb{N}_0}$ be a sequence of distinct positive real numbers satisfying

$$\sum_{n=0}^{\infty} \frac{(\lambda_n - \frac{1}{4}\pi^2 n^2)_+}{n^2} < \infty. \tag{2.1}$$

Let m_1, m_2 be the m-functions for two operators $H_j = -\frac{d^2}{dx^2} + q_j$ in $L^2((0,1))$ with boundary conditions

$$u'(1) + h_1^{(j)} u(1) = 0$$

and $h_1^{(j)} \in \mathbb{R},\ j = 1, 2$. Suppose that $m_1(\lambda_n) = m_2(\lambda_n)$ for all $n \in \mathbb{N}_0$. Then $m_1 = m_2$ (and hence $q_1 = q_2$ a.e. on $[0,1]$ and $h_1^{(1)} = h_1^{(2)}$). □

Remarks. (1) In our examples, $\lambda_n \sim \pi^2 n^2 + C$ as $n \to \infty$ (cf. (3.1)), so (2.1) is satisfied, for instance, by considering two distinct spectra of $H(h_0)$.

(2) We allow the case $m_1(\lambda_n) = m_2(\lambda_n) = \infty$.

As a preliminary result, we note the following.

Lemma 2.2. Suppose $\{\lambda_n\}_{n\in\mathbb{N}_0}$ is a sequence of positive real numbers satisfying (2.1) and

$$\sum_{n=0}^{\infty} \lambda_n^{-1} < \infty. \tag{2.2}$$

Define $f(z) := \prod_{n=0}^{\infty}(1 - \frac{z}{\lambda_n})$; then

$$\varlimsup_{\substack{|y|\to\infty \\ y\in\mathbb{R}}} \frac{|y|^{1/2} \sinh(2|y|^{1/2})}{|f(iy)|} < \infty. \tag{2.3}$$

□

Proof. Let $y \in \mathbb{R}$. Then $\sinh(2|y|^{1/2})/|y|^{1/2} = |\sin(2i|y|^{1/2})/|y|^{1/2}|$ and

$$\frac{\sin(2\sqrt{z})}{2\sqrt{z}} = \prod_{n=1}^{\infty}\left(1 - \frac{4z}{\pi^2 n^2}\right),$$

so (2.3) becomes

$$\varlimsup_{|y|\to\infty} \frac{|y|}{1+\frac{|y|}{\lambda_0}} \prod_{n=1}^{\infty}\left[\frac{(1+\frac{4|y|}{\pi^2 n^2})}{(1+\frac{|y|}{\lambda_n})}\right] < \infty \tag{2.4}$$

using $2^{-1/2}(1+|x|) \le (1+x^2)^{1/2} \le (1+|x|)$.

If $0 \le a \le b$, then

$$\left(\frac{1+a|y|}{1+b|y|}\right) \le 1,$$

and if $a > b > 0$, then

$$\frac{(1+a|y|)}{1+b|y|} = 1 + \frac{(a-b)|y|}{1+b|y|} \le 1 + \frac{a-b}{b} = \frac{a}{b},$$

$$\prod_{n=1}^{\infty} \frac{(1+\frac{4|y|}{\pi^2 n^2})}{(1+\frac{|y|}{\lambda_n})} \le \prod_{n:\lambda_n > \frac{1}{4}\pi^2 n^2} \frac{4\lambda_n}{\pi^2 n^2} = \prod_{n=1}^{\infty}\left[1 + \frac{(\lambda_n - \frac{1}{4}\pi^2 n^2)_+}{\frac{1}{4}\pi^2 n^2}\right] < \infty$$

if (2.1) holds. ■

Proof of Theorem 2.1. We follow the arguments in [5], [6] fairly closely. One can write $m_j(z) = Q_j(z)/P_j(z),\ j = 1, 2$, where

(1) P_j, Q_j are entire functions satisfying

$$|P_j(z)| \le C\exp(\sqrt{|z|}), \tag{2.5a}$$

$$|Q_j(z)| \le C(1+\sqrt{|z|})\exp(\sqrt{|z|}); \tag{2.5b}$$

(2)

$$m_j(z) = \pm i\sqrt{z} + o(1) \text{ as } z \to \pm i\infty. \tag{2.6}$$

(We use the square root branch with $\operatorname{Im}(\sqrt{z}) \ge 0$.)

Suppose $m_1 \ne m_2$. Then $P_2(z)Q_1(z) - P_1(z)Q_2(z) := H(z)$ is an entire function of order at most $1/2$ and not identically zero. Since $H(\lambda_n) = 0$, we conclude that $\sum_{n\in\mathbb{N}_0} \lambda_n^{-a} < \infty$ if $a > 1/2$. In particular, (2.2) holds, and we can define $f(z) = \prod_{n=0}^{\infty}(1 - z/\lambda_n)$. Next, define

$$G(z) := \frac{H(z)}{f(z)} = \frac{P_1(z)P_2(z)}{f(z)}(m_1(z) - m_2(z)). \tag{2.7}$$

Since $H(\lambda_n) = 0$, $G(z)$ is an entire function. By (2.3),

$$\overline{\lim_{|y|\to\infty}} \frac{|y|^{1/2} \exp(2|y|^{1/2})}{|f(iy)|} < \infty.$$

So, by (2.5) and (2.6),

$$|G(iy)| \le \frac{\exp(2|y|^{1/2})}{f(iy)} |m_1(iy) - m_2(iy)| = o(|y|^{-1/2})$$

goes to zero as $|y| \to \infty$. The Phragmén-Lindelöf argument of [6] then yields the contradiction $G(z) \equiv 0$; that is, $m_1 = m_2$. ■

Remark. The above yields $o(|y|^{-1/2})$ even though $o(1)$ would have been sufficient. We have thrown away half a zero. That means one can prove the following result.

Theorem 2.2. Let $\{\lambda_n\}_{n\in\mathbb{N}_0}$ and $\{\mu_n\}_{n\in\mathbb{N}_0}$ be two sequences of real numbers satisfying

$$\sum_{n=0}^{\infty} \frac{(\lambda_n - \pi^2 n^2)_+}{n^2} < \infty \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{(\mu_n - \pi^2 n^2)_+}{n^2} < \infty, \tag{2.8}$$

with $\mu_m \neq \lambda_n$ for all $m, n \in \mathbb{N}_0$. Let m_1, m_2 be the m-functions for two operators $H_j = -(d^2/dx^2) + q_j$, $j = 1, 2$ in $L^2((0,1))$ with boundary conditions

$$u'(1) + h_1^{(j)} u(1) = 0$$

and $h_1^{(j)} \in \mathbb{R}$, $j = 1, 2$. Suppose that $m_1(z) = m_2(z)$ for all z in $\{\lambda_n\}_{n=0}^{\infty} \cup \{\mu_n\}_{n=0}^{\infty}$ except perhaps for one. Then $m_1 = m_2$ (and hence $q_1 = q_2$ a.e. on $[0,1]$ and $h_1^{(1)} = h_1^{(2)}$). □

By scaling, one sees that the following analog of Theorem 2.1 holds (there is also an analog of Theorem 2.2).

Theorem 2.3. Let $a < b$ and let $\{\lambda_n\}_{n\in\mathbb{N}_0}$ be a sequence of distinct positive real numbers satisfying

$$\sum_{n=0}^{\infty} \frac{\left(\lambda_n - \frac{\pi^2 n^2}{4(b-a)^2}\right)_+}{n^2} < \infty. \tag{2.9}$$

Let m_1, m_2 be the m-functions for two operators $H_j = -(d^2/dx^2) + q_j$, $j = 1, 2$ in $L^2((a,b))$ with boundary conditions (1.3) at $x = a$ and

$$u'(b) + h_b^{(j)} u(b) = 0,$$

where $h_b^{(j)} \in \mathbb{R}$, $j = 1, 2$. Suppose that $m_1(\lambda_n) = m_2(\lambda_n)$ for all $n \in \mathbb{N}_0$. Then $m_1 = m_2$ (and hence $q_1 = q_2$ a.e. on $[a,b]$ and $h_b^{(1)} = h_b^{(2)}$). □

3 Whole interval results

Fix $h_1 \in \mathbb{R}$, let $H(h_0)$ be the operator on $L^2((0,1))$ with $u'(1)+h_1u(1) = 0$ and $u'(0)+h_0u(0) = 0$ boundary conditions, and denote by $\lambda_n(h_0)$ the corresponding eigenvalues of $H(h_0)$. Then, for $h_0 \in \mathbb{R}$, it is known (see, e.g., the references in [6]) that

$$\lambda_n = (n\pi)^2 + 2(h_1 - h_0) + \int_0^1 q(x)\,dx + o(1) \text{ as } n \to \infty \tag{3.1}$$

and for $h_0 = \infty$,

$$\lambda_n = [(n + \tfrac{1}{2})\pi]^2 + 2h_1 + \int_0^1 q(x)\,dx + o(1) \text{ as } n \to \infty. \tag{3.2}$$

To say that $H(h_0)$ has eigenvalue λ is equivalent to $m(\lambda) = -h_0$. Thus, Theorem 2.1 implies the following.

Theorem 3.1. Let $H_1(h_0), H_2(h_0)$ be associated with two potentials q_1, q_2 on $[0,1]$ and two potentially distinct boundary conditions $h_1^{(1)}, h_1^{(2)} \in \mathbb{R}$ at $x = 1$. Suppose that $\{(\lambda_n, h_0^{(n)})\}_{n\in\mathbb{N}_0}$ is a sequence of pairs with $\lambda_0 < \lambda_1 < \cdots \to \infty$ and $h_0^{(n)} \in \mathbb{R} \cup \{\infty\}$ so that both $H_1(h_0^{(n)})$ and $H_2(h_0^{(n)})$ have eigenvalues at λ_n. Suppose that (2.1) holds. Then $q_1 = q_2$ a.e. on $[0,1]$ and $h_1^{(1)} = h_1^{(2)}$. □

Given (3.1), (3.2) we immediately have Borg's theorem [1] as a corollary (this is essentially the usual proof), but more is true. For example, by using Theorem 2.2, one infers the following.

Corollary 3.2 [1]. Fix $h_0^{(1)}, h_0^{(2)} \in \mathbb{R}$. Then all the eigenvalues of $H(h_0^{(1)})$ and all the eigenvalues of $H(h_0^{(2)})$, save one, uniquely determine q a.e. on $[0,1]$. □

Corollary 3.3. Let $h_0^{(1)}, h_0^{(2)}, h_0^{(3)} \in \mathbb{R}$ and denote by $\sigma_j = \sigma(H(h_0^{(j)}))$ the spectra of $H(h_0^{(j)})$, $j = 1,2,3$. Assume $S_j \subseteq \sigma_j$, $j = 1,2,3$ and suppose that for all sufficiently large $\lambda_0 > 0$, we have

$$\#\{\lambda \in \{S_1 \cup S_2 \cup S_3\} \quad \text{with} \quad \lambda \le \lambda_0\} \ge \tfrac{2}{3}\#\{\lambda \in \{\sigma_1 \cup \sigma_2 \cup \sigma_3\} \quad \text{with} \quad \lambda \le \lambda_0\} - 1.$$

Then q is uniquely determined a.e. on $[0,1]$. □

In particular, two-thirds of three spectra determine q.

4 Updating m

We are now able to understand why partial information on q – knowing it on $[0,a]$ – lets us get away with less information on eigenvalues, a phenomenon originally discovered by

Hochstadt-Lieberman [9] in the special case where $a = 1/2$. We note that $m(z, x)$ satisfies the Ricatti-type equation

$$m'(z, x) = q(x) - z - m^2(z, x). \tag{4.1}$$

If we know that λ is an eigenvalue of $H(h_0)$, then $m(\lambda, 0) = -h_0$. If we know q on $[0, a]$, we can use (4.1) to compute $m(\lambda, a) := -h_a$ and so infer that λ is an eigenvalue of $H(h_a)$, the operator in $L^2((a, 1))$. By Theorem 2.3, this means we only need a lower density of eigenvalues of the various $H(h_a)$. A typical result is the following theorem.

Theorem 4.1. Let σ_N and σ_D be the eigenvalues of $H(h_0 = 0)$ and $H(h_0 = \infty)$, respectively. Let $S_N \subseteq \sigma_N$, $S_D \subseteq \sigma_D$. Fix $a \in (0, 1)$. Suppose for $\lambda_0 > 0$ sufficiently large that

$$\#\{\lambda \in \{S_N \cup S_D\} \quad \text{with} \quad \lambda \leq \lambda_0\} \geq (1 - a)\#\{\lambda \in \{\sigma_N \cup \sigma_D\} \quad \text{with} \quad \lambda \leq \lambda_0\}.$$

Then S_N, S_D, and q on $[0, a]$ uniquely determine q a.e. on $[0, 1]$. □

This follows immediately from the updating idea. For example, if $a = 3/4$, we can recover Theorem 1.3 of [6] (it is essentially a reworking of the proof in [6]); but for $a \in (0, 1/2)$, the result is new and implies, for example, that q on $[0, 1/4]$, all the Neumann eigenvalues, and half the Dirichlet eigenvalues, uniquely determine q a.e. on $[0, 1]$.

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