POINT MASS PERTURBATIONS OF SPECTRAL MEASURES

RAFAEL DEL RIO, LUIS O. SILVA AND JULIO H. TOLOZA

Using a generalization of the moment problem and the extremal properties of spectral measures corresponding to the selfadjoint extensions of a regular symmetric operator, we study point mass perturbations of spectral measures. We obtain general results for a wide class of operators and apply them to the analysis of point mass perturbations of spectral measures pertaining to Bessel and generalized Schrödinger operators.

1. Introduction

The motivation of this work stems from the intriguing fact that very small perturbations of a measure may destroy the density of polynomials in the space of square integrable functions with respect to that measure. More precisely: one can find a Borel measure ρ such that the polynomials are dense in $L_2(\mathbb{R}, \rho)$, but they are no longer dense in $L_2(\mathbb{R}, \tilde{\rho})$ when $\tilde{\rho}$ is obtained by adding a point mass to ρ, that is,

$$(1) \qquad \tilde{\rho} = \rho + \delta_\lambda, \quad \lambda \notin \operatorname{supp} \rho,$$

where δ_λ denotes a discrete measure which gives weight only to the point λ. This instability phenomenon pertains to the properties of solutions to the classical Hamburger moment problem. In this context, the phenomenon is formulated as follows. If ρ is an N-extremal solution to an indeterminate moment problem, then $\tilde{\rho}$ is a non-N-extremal solution to a moment problem (see [1, Theorem 2.3.2] and [36, Proposition 4.1]). Remarkably, the theory of the Hamburger moments is the theory of Jacobi operators, i.e., operators in $l_2(\mathbb{N})$ generated by semiinfinite tridiagonal symmetric matrices (see Section 6.1). In this context, the mentioned phenomenon is paraphrased as follows: if ρ is the spectral measure of a canonical selfadjoint extension of a nonselfadjoint Jacobi operator, i.e., a selfadjoint restriction of its adjoint (see Section 2.1), then there is no canonical selfadjoint extension of any Jacobi operator having $\tilde{\rho}$ as its spectral measure.

This paper is devoted to studying the situation illustrated above in a setting that allows us to consider not only nonselfadjoint Jacobi operators, but a wide class of regular symmetric operators with deficiency indices (1, 1), including regular and certain singular Schrödinger operators. Our approach is based on Krein's theory of symmetric operators [19; 20; 21], the theory of generalized extensions due to Naĭmark [31] and a generalization of the moment problem originally proposed by Livsič [27]. This theoretical

scaffold allows us to elucidate the structure of the set of spectral measures and to prove several results on point mass perturbations of spectral measures presented in Section 5. The central theorem (Theorem 5.4) is general enough to be applied to the analysis of point mass perturbation of spectral measures related to generalized Schrödinger operators with measures and Bessel operators. This analysis led to Theorems 6.2, 6.4, and 6.5.

The spectrum of a regular Schrödinger operator, defined by a selfadjoint boundary condition (excluding Dirichlet) at the left endpoint and a certain selfadjoint boundary condition at the right endpoint, has the following asymptotic behavior:

$$\lambda_n = cn^2 + O(1) \text{ as } n \to \infty, \tag{2}$$

(see [16; 32]). From this formula, by a straightforward argument (see Remark 6.3), one concludes that if ρ is the spectral measure of a regular Schrödinger operator with these selfadjoint boundary conditions, then $\tilde{\rho}$, defined as in (1), cannot be the spectral measure of any regular Schrödinger operator with the same selfadjoint boundary condition at the left endpoint and any selfadjoint boundary condition at the right endpoint no matter which potential function is considered. It is worth remarking that the results of Section 5 provide a way for dealing with point mass perturbation of spectral measures for a wide range of operators even in the case when asymptotics of the kind of (2) are not available. This is illustrated in Sections 6.2 and 6.3.

Let us elaborate on the theoretical framework developed in this paper which is of interest in itself irrespective of the results given in Sections 4 and 6. The classical theory of moments tells us that to account for all solutions to the moment problem, one has to recur to Naĭmark's theory of generalized extensions [30; 31] since the solutions are the generalized spectral measures (see Definition 2.5) of the corresponding Jacobi operator. Here we remark that, despite the classicality of the theory, relatively recent developments [9; 46] make use of [31] to construct solutions to the classical moment problem with the measure having certain predefined properties. Our approach makes use not only of Naĭmark's theory, but also Krein's theory of symmetric operators with deficiency indices (1,1) [19; 20; 21] to study the set of measures associated with an arbitrary regular symmetric operator not necessarily entire [14]. This allows us to characterize the spectral measures associated with the operator (Theorem 3.4) and describe the extremal properties of measures related to this operator's selfadjoint extensions (Theorem 3.6). The next step in our method is the use of the functional model for regular symmetric operators [39; 40] to map any regular symmetric operator into the multiplication operator in a de Branges space. This reformulation leads us directly to the generalized moment problem.

There are various ways of generalizing the moment problem. Here, we have in mind a proposal of generalization that goes back to Livšic [27] for whom the moment problem consists in finding a measure ρ such that the inner product in a Hilbert space of functions is expressed through the inner product in $L_2(\mathbb{R}, \rho)$. A related setting of the generalized moment problem is found in the introduction of [45]. As pointed out in [35, Chapter 6], de Branges treated this problem using a different terminology when the Hilbert space of functions is a de Branges space [8]. In this work, we also consider the generalization of [27] with the underlying Hilbert space being a de Branges space, but in contrast to [8] and [35, Chapter 6], we focus on the extremal properties of the solutions to the generalized moment problem.

Let us outline the material of this paper. Section 2 presents classical results that are included for the sake of completeness and to introduce the notation. Section 3 studies the extremal properties of

measures associated with regular symmetric operators. A short survey on de Branges space theory and the generalized moment problem are the subject of Section 4. In this section, we also present the functional model for regular symmetric operators by means of which we can use the theory of de Branges spaces to study our class of operators. In Section 5, we prove the main results which account for how the properties of spectral measures change when adding a mass at a single point. Here we also touch upon the fact that certain functions which are dense in $L_2(\mathbb{R}, \rho)$ are no longer dense in $L_2(\mathbb{R}, \tilde{\rho})$ (see (1)). These results may have several applications to the inverse spectral analysis of operators as is shown in [36] for Jacobi operators. Finally, in Section 6, we illustrate the results of this work. The first example is used to put our results in the context of the classical moment problem. The other two examples present nontrivial results pertaining to differential operators which include regular and singular Schrödinger operators.

2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space. The inner product in it is denoted by $\langle \cdot, \cdot \rangle$, where we agree that the left-hand side argument is antilinear. We denote by $\oplus$, $\ominus$, and $\dotplus$ the orthogonal sum, the orthogonal complement, and the direct sum of linear spaces, respectively.

Let A be a linear closed symmetric operator, not necessarily densely defined. If A is not densely defined, then A^* is a multivalued linear operator (linear relation). In any case, the symmetry condition, $A \subset A^*$, remains valid (since in the theory of linear relations, one identifies A and A^* with the corresponding subspaces of $\mathcal{H} \oplus \mathcal{H}$). The deficiency indices $n_+(A)$ and $n_-(A)$ of A are defined as the dimension of $\mathcal{H} \ominus \mathrm{ran}(A - zI)$, when $z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ and $z \in \mathbb{C}_- := \{z \in \mathbb{C} : \mathrm{Im}\, z < 0\}$, respectively. If A has equal deficiency indices, then it has canonical selfadjoint extensions (that is, selfadjoint restrictions of A^*), with some of these extensions being proper multivalued linear operators if $\mathrm{dom}(A)$ is not dense in $\mathcal{H}$ [15, Proposition 5.4].

What follows is a brief revision of Naĭmark's theory of selfadjoint extensions to a larger space [31; 30] (see also [2, Appendix 1] and [45]), as well as some results from Krein's theory on regular symmetric operators [14; 21].

2.1. Generalized resolvents.

Definition 2.1. Consider a closed symmetric operator A. Let A^+ be a selfadjoint extension of A in a Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$ ($\mathcal{H}^+ = \mathcal{H}$ is allowed). For $w \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \mathrm{spec}(A^+)$, define

$$(3) \qquad V_{A^+}(w, z) := (A^+ - wI)(A^+ - zI)^{-1} = I + (z - w)R^+(z),$$

where R^+ is the resolvent of A^+ (see [14, Chapter 1, §2.1]).

From the first resolvent identity (see [6, Chapter 3, §7]), one verifies that

$$(4) \qquad V_{A^+}(w, z) = V_{A^+}(z, w)^{-1} \quad \text{and} \quad V_{A^+}(w, z)V_{A^+}(z, v) = V_{A^+}(w, v)$$

hold true for any $v, w, z \in \mathbb{C} \setminus \mathrm{spec}(A^+)$. Also, it is straightforward to establish that

$$(5) \qquad V_{A^+}(w, z)^* = V_{A^+}(\bar{w}, \bar{z}).$$

The following assertion arises by combining the first identity in (4) with (5) (see [14, Chapter 1, §2.1]).

Proposition 2.2. *Let $z \in \mathbb{C}$ and $w \in \mathbb{C}_+$. For any selfadjoint extension A^+ of A such that $z \notin \mathrm{spec}(A^+)$, the operator $V_{A^+}(w, z)$ maps $\mathcal{H}^+ \ominus \mathrm{ran}(A - \bar{w}I)$ injectively onto $\mathcal{H}^+ \ominus \mathrm{ran}(A - \bar{z}I)$.*

Definition 2.3. A scalar function with domain of analyticity containing $\mathbb{C}_+$ is a Herglotz function whenever $\mathrm{Im}\, z > 0 \Longrightarrow \mathrm{Im}\, f(z) \geq 0$. By this definition, any real constant is a Herglotz function. Denote by $\mathfrak{R}$ the union of the class of Herglotz functions and the constant ∞.

Definition 2.4. Denote by P^+ the orthogonal projector of $\mathcal{H}^+$ onto $\mathcal{H}$ and by $R^+(z)$ the resolvent of A^+. The operator

$$R(z) := P^+ R^+(z) \restriction_{\mathcal{H}}, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

is a generalized resolvent of A [31] (see also [2, Appendix 1] and [45, §3]).

Fix a canonical selfadjoint extension of a closed symmetric operator A with deficiency indices $n_+(A) = n_-(A) = 1$ and denote it by A_∞. For $w \in \mathbb{C}_+$ and $z \in \mathbb{C} \setminus \mathrm{spec}(A_\infty)$, define

$$(6) \qquad \qquad \psi(z) := V_{A_\infty}(w, z)\psi,$$

where $\psi \in \ker(A^* - wI) \setminus \{0\}$ (see [2, Appendix 1, §4, Equation 2']).

Let $R_\infty(z)$ be the resolvent of A_∞. Given $\tau \in \mathfrak{R}$, define

$$(7) \qquad \qquad R_\tau(z) := R_\infty(z) - \frac{\langle \psi(\bar{z}), \cdot \rangle \psi(z)}{\tau(z) + Q(z)},$$

where

$$Q(z) := i\, \mathrm{Im}\, w + (z - w)\langle \psi(\bar{z}), \psi(w) \rangle$$

is the so-called Q-function (see [2, Appendix 1, Equation 8]). $R_\tau(z)$ is the resolvent of a canonical selfadjoint extension of A if and only if $\tau(z) \equiv c \in \mathbb{R} \cup \{\infty\}$. Moreover, once a canonical selfadjoint extension A_∞ is fixed, (7) induces a bijection between $\mathfrak{R}$ and the set of all generalized resolvents of A (see [10, Theorem 6.2], also [22; 23; 24; 31; 44]). Equation (7) is known as the Krein–Naĭmark formula.

Definition 2.5. Let E_{A^+} denote the spectral family of A^+ given by the spectral theorem. Let $\mathfrak{X}$ denote the σ-algebra of Borel sets on $\mathbb{R}$. Given a selfadjoint extension A^+ of a symmetric operator A, the map

$$F_{A^+}(\partial) := P^+ E_{A^+}(\partial) \restriction_{\mathcal{H}}, \quad \partial \in \mathfrak{X},$$

is called a generalized spectral family of the operator A. Furthermore, given $\phi \in \mathcal{H} \setminus \{0\}$, we define

$$(8) \qquad \qquad \sigma_{A^+,\phi}(\partial) := \langle \phi, F_{A^+}(\partial)\phi \rangle_{\mathcal{H}}, \quad \partial \in \mathfrak{X}.$$

We call $\sigma_{A^+,\phi}$ the generalized spectral measure of A associated with ϕ and A^+.

Note that

$$\sigma_{A^+,\phi}(\partial) = \langle \phi, E_{A^+}(\partial)\phi \rangle_{\mathcal{H}^+}, \quad \partial \in \mathfrak{X},$$

so $\sigma_{A^+,\phi}$ is also the spectral measure of the selfadjoint extension A^+ associated with $\phi \in \mathcal{H}$ and therefore it is a nonnegative finite scalar measure. On the basis of [2, Appendix I, Theorem 2], the next result holds true.

Proposition 2.6. *F is a generalized spectral family of A* (*that is, $F = F_{A^+}$ for some selfadjoint extension A^+ of A*) *if and only if the operator-valued function $F(t) := F(-\infty, t)$ ($t \in \mathbb{R}$) satisfies, for any $\phi \in \mathcal{H}$, the following conditions*:

(i) *The function $\langle \phi, F(t)\phi \rangle$ does not decrease when t increases.*

(ii) *$F(t)\phi$ is left continuous.*

(iii) *$F(t)\phi \xrightarrow[t \to -\infty]{} 0$ and $F(t)\phi \xrightarrow[t \to \infty]{} \phi$.*

(iv) *If $\phi \in \mathrm{dom}(A)$, then*

$$\|A\phi\|^2 = \int_{\mathbb{R}} t^2 d\langle \phi, F(t)\phi \rangle \quad and \quad A\phi = \int_{\mathbb{R}} t\, dF(t)\phi.$$

The next statement is proven on the basis of Proposition 2.6 (see [2, Appendix 1, pp. 141–143]).

Proposition 2.7. *There is a bijection between $\mathfrak{R}$ and the generalized spectral family of any symmetric operator with deficiency indices $(1, 1)$. The bijection is given by the combination of (7) and the identity*

$$(9) \qquad \langle \phi, R(z)\phi \rangle_{\mathcal{H}} = \langle \phi, R^+(z)\phi \rangle_{\mathcal{H}^+} = \int_{\mathbb{R}} \frac{d\sigma_{A^+,\phi}(t)}{t - z}.$$

Moreover, the same bijection gives a one-to-one map between $\mathbb{R} \cup \{\infty\}$ and the spectral family corresponding to canonical selfadjoint extensions of A.

2.2. Regular symmetric operators and gauges.

Definition 2.8. A closed symmetric nonselfadjoint operator is said to be completely nonselfadjoint if it is not a nontrivial orthogonal sum of a symmetric and a selfadjoint operators.

Definition 2.9. The complex number z is in the set of points of regular type of a closed operator T when there exists $c_z > 0$ such that

$$\|(T - zI)\phi\| \geq c_z \|\phi\| \quad \text{for all} \quad \phi \in \mathrm{dom}(T).$$

The complement of the set of points of regular type of T is called the spectral kernel of T.

Definition 2.10. A closed operator T is regular when the set of points of regular type is the whole complex plane.

The notion of regularity just introduced should not be confused with the regularity of operators generated by differential expressions (see Introduction and Section 6.2)

A regular, closed symmetric operator is necessarily completely nonselfadjoint since regularity implies that the spectral kernel is empty and, therefore, the operator cannot have selfadjoint parts. On the other hand, there are many completely nonselfadjoint operators that are not regular [42, §2]. We note that the dimension of $\ker(A^* - zI)$ is constant for all $z \in \mathbb{C}$ whenever A is regular.

Definition 2.11. Let $\mathscr{S}(\mathcal{H})$ denote the set of regular, closed symmetric operators in $\mathcal{H}$ with deficiency indices $n_+(A) = n_-(A) = 1$.

The next assertion is well known for densely defined symmetric operators (see [6, Chapter 9, §3, Theorem 9] and also [3; 11; 14]). A proof for the case under discussion in this work can be found in [40, Proposition 2.4].

Proposition 2.12. *Let A be in $\mathcal{S}(\mathcal{H})$. The following statements are true*:

(i) *The spectrum of every canonical selfadjoint extension of A consists solely of isolated eigenvalues of multiplicity one.*

(ii) *Every real number is part of the spectrum of one, and only one, canonical selfadjoint extension of A.*

(iii) *The spectra of the canonical selfadjoint extensions of A are pairwise interlaced.*

Note that item (i) above means that every selfadjoint extension of A is a simple operator [2, §69]. Also, item (iii) above (together with the fact that under extensions the spectral kernel does not decrease) implies that A is regular if and only if the spectra of any two selfadjoint extensions of A do not intersect.

For any $A \in \mathcal{S}(\mathcal{H})$, let $\kappa \in \mathcal{H}$ be such that

$$(10) \qquad\qquad \mathcal{H} = \operatorname{ran}(A - z_0 I) \dotplus \operatorname{span}\{\kappa\},$$

for some $z_0 \in \mathbb{C}$. For this vector κ, consider the set

$$(11) \qquad \{z \in \mathbb{C} : \kappa \in \operatorname{ran}(A - zI)\} = \{z \in \mathbb{C} : \mathcal{H} \neq \operatorname{ran}(A - zI) \dotplus \operatorname{span}\{\kappa\}\}.$$

The following result is proven in [39, Theorem 2.2]. It was first formulated by Krein without proof in [21, Theorem 8].

Proposition 2.13. *For every $A \in \mathcal{S}(\mathcal{H})$, there exists κ satisfying (10) such that the corresponding set (11) does not intersect the real line.*

Definition 2.14. We call the vector κ, whose existence is established in Proposition 2.13, a *spectral gauge* of $A \in \mathcal{S}(\mathcal{H})$.

Lemma 2.15. *A spectral gauge of $A \in \mathcal{S}(\mathcal{H})$ is a generating vector (as defined in [2, §69]) for any canonical selfadjoint extension of A.*

Proof. Let A_γ be any canonical selfadjoint extension of A and κ a spectral gauge of A. Since κ is not in $\operatorname{ran}(A - xI)$ for any $x \in \mathbb{R}$, κ has a nonzero projection onto the one-dimensional space $\ker(A^* - xI)$. Because $\ker(A^* - xI) = \ker(A_\gamma - xI)$ whenever $x \in \operatorname{spec}(A_\gamma)$, it follows that κ has a nonzero projection onto every eigenspace of A_γ since its spectrum is discrete. $\qquad\square$

The following assertion is stated in [21, Theorem 1] and proven in [14, Theorem 1.2.5].

Proposition 2.16. *Let κ be a spectral gauge of $A \in \mathcal{S}(\mathcal{H})$ and $a, b \in \mathbb{R}$ be such that $a < b$. If F is a generalized spectral family of A, then*

$$(12) \qquad\qquad F([a, b])\phi = \int_a^b \frac{\langle \psi(t), \phi \rangle}{\langle \psi(t), \kappa \rangle} \, dF(t)\kappa$$

for any $\phi \in \mathcal{H}$ and ψ given in (6).

3. Extremality of measures

Definition 3.1. Consider $A \in \mathscr{S}(\mathcal{H})$ and let κ be a spectral gauge of A. Recalling (8), define

$$\mathcal{V}_\kappa(A) := \{\sigma = \sigma_{A^+,\kappa} : A^+ \text{ is selfadjoint extension of } A\}.$$

Inside $\mathcal{V}_\kappa(A)$, we single out the set of extremal measures

$$\mathcal{V}_\kappa^{\text{ext}}(A) := \{\sigma = \sigma_{A^+,\kappa} : A^+ \text{ is canonical selfadjoint extension of } A\}.$$

The reason for using word "extremal" above will be explained in Theorem 3.6 below. In the context of this definition, Proposition 2.7 implies the following.

Corollary 3.2. *For any spectral gauge κ of $A \in \mathscr{S}(\mathcal{H})$, there is a bijection between $\mathfrak{R}$ and $\mathcal{V}_\kappa(A)$. Moreover, this bijection gives a one-to-one mapping between $\mathbb{R} \cup \{\infty\}$ and $\mathcal{V}_\kappa^{\text{ext}}(A)$.*

The bijection referred above depends on the choice of A_∞ (see Section 2.1), but regardless of this choice, all measures arising from spectral families corresponding to canonical selfadjoint extensions of A are obtained when τ runs through $\mathbb{R} \cup \{\infty\}$. Note that, as a consequence of Proposition 2.12, one has:

Corollary 3.3. *Let κ be a spectral gauge of $A \in \mathscr{S}(\mathcal{H})$. All measures in $\mathcal{V}_\kappa^{\text{ext}}(A)$ are discrete and their supports are pairwise disjoint. For any $x \in \mathbb{R}$, there is a measure $\sigma \in \mathcal{V}_\kappa^{\text{ext}}(A)$ such that $\sigma\{x\} > 0$.*

The next statement can be found in [14, Chapter 2, §7.1] and the first part of it in [2, Appendix I, §4] (see [20, Theorem 6] and [19]).

Theorem 3.4. *Consider $A \in \mathscr{S}(\mathcal{H})$. Given $\tau \in \mathfrak{R}$, let $R_\tau(z)$ be a generalized resolvent as in (7). If κ is a spectral gauge of A, then*

$$\langle \kappa, R_\tau(z)\kappa \rangle = -\frac{\tau(z)A(z) - C(z)}{\tau(z)B(z) - D(z)},$$

where $A(z), B(z), C(z), D(z)$ are meromorphic functions on $\mathbb{C}$, analytic on $\mathbb{R}$.

Proof. Reasoning as in [14, Chapter 42, §7.1], one obtains the stated formula by inserting the right-hand side expression from (7) into $\langle \kappa, R_\tau(z)\kappa \rangle$ and considering the functions:

$$(13) \qquad\qquad A(z) := -\frac{\langle \kappa, R_\infty(z)\kappa \rangle}{\langle \psi(\bar{z}), \kappa \rangle},$$

$$(14) \qquad\qquad B(z) := \frac{1}{\langle \psi(\bar{z}), \kappa \rangle},$$

$$(15) \qquad\qquad C(z) := -Q(z)A(z) - \langle \kappa, \psi(z) \rangle,$$

$$(16) \qquad\qquad D(z) := -Q(z)B(z),$$

where ψ is given in (6). One verifies from their definition that $A(z), B(z), D(z)$ can be analytically extended to any point z outside the set given in (11). The function $C(z)$ has a priori poles in $\text{spec}(A_\infty)$, but by an argument along the lines of [14, Chapter 2, §7.3 and 7.4], one concludes that $C(z)$ can be analytically extended to any point z outside the set given in (11). $\qquad\square$

Remark 3.5. For Jacobi operators, the functions $A(z), B(z), C(z), D(z)$ coincide with the ones given in [1, Equation 2.28] and [43, Theorem 4.9]. Note that, for the Herglotz function $-1/\tau$, [43, Equation 4.36] is indeed an expression for $\langle \kappa, R_{-1/\tau}(z)\kappa \rangle$.

Reasoning along the lines of the proof of [43, Theorem 4.17], one obtains the following assertion from Theorem 3.4.

Theorem 3.6. *Let κ be a spectral gauge of $A \in \mathcal{S}(\mathcal{H})$ and λ a real number. If $\sigma \in \mathcal{V}_\kappa(A) \setminus \mathcal{V}_\kappa^{\text{ext}}(A)$ is such that $\sigma\{\lambda\} > 0$, then there exists $\tilde\sigma \in \mathcal{V}_\kappa^{\text{ext}}(A)$ such that $\tilde\sigma\{\lambda\} > \sigma\{\lambda\}$.*

4. On de Branges spaces and a generalization of the moment problem

There are various ways of generalizing the moment problem. We use the generalization proposed by Livšič [27] (see also [45]). According to [27], the generalized moment problem consists in finding a measure ρ such that the inner product in a Hilbert space of functions is expressed through the inner product in $L_2(\mathbb{R}, \rho)$. Our choice for the Hilbert space of functions is a de Branges space. In [35, Chapter 6], the same choice is done for the generalized moment problem and the treatment is made by means of canonical systems whose theory contains the one of de Branges spaces (see [37]). In contrast to [35, Chapter 6], our final goal is the extremal properties of measures.

Definition 4.1. A Hilbert space of entire functions $\mathcal{B}$ is a de Branges space if and only if, for any function $f(z)$ in $\mathcal{B}$, the following conditions holds:

(A1) For all $w \in \mathbb{C}$, the linear functional $f(\cdot) \mapsto f(w)$ is continuous;

(A2) for every nonreal zero w of $f(z)$, the function $f(z)(z - \bar{w})(z - w)^{-1}$ belongs to $\mathcal{B}$ and has the same norm as $f(z)$;

(A3) the function $f^\#(z) := \overline{f(\bar{z})}$ also belongs to $\mathcal{B}$ and has the same norm as $f(z)$.

Alternatively, a de Branges space can be defined in terms of an Hermite–Biehler function, that is, an entire function $e(z)$ such that $|e(z)| > |e(\bar{z})|$ for all $z \in \mathbb{C}_+$ [25, Chapter 7]. In that case, the de Branges space is sometimes denoted $\mathcal{B}(e)$ and one has

$$(17) \qquad \langle f, g \rangle_\mathcal{B} = \int_\mathbb{R} \frac{\overline{f(x)} g(x)}{|e(x)|^2} \, dx.$$

(see [8, §19], [34, Theorem 2.2], [42]). The right-hand side of (17) is justified by the fact that if x is a real zero of $e(z)$, then x is a zero of greater or equal multiplicity for any function in $\mathcal{B}(e)$. We note that a given de Branges space can be obtained by different Hermite–Biehler functions [7, Theorem 1].

Definition 4.2. Given a de Branges space $\mathcal{B}$, the operator S of multiplication by the independent variable is defined by

$$\text{dom}(S) = \{f(z) \in \mathcal{B} : zf(z) \in \mathcal{B}\}; \quad (Sf)(z) = zf(z), \quad f(z) \in \text{dom}(S).$$

It is noteworthy that the operator S is in $\mathcal{S}(\mathcal{B})$ [17, Proposition 4.2, Corollary 4.7]. Moreover, to any operator A in the class $\mathcal{S}(\mathcal{H})$, there corresponds a de Branges space such that the operator of multiplication in it is unitarily equivalent to A. To elucidate this, we introduced below the so-called functional model for operators in $\mathcal{S}(\mathcal{H})$.

An involution (conjugation) $J : \mathcal{H} \to \mathcal{H}$ is an antilinear map satisfying

$$J^2 = I \quad \text{and} \quad \langle J\phi, J\psi \rangle = \langle \psi, \phi \rangle$$

for all ϕ, $\psi \in \mathcal{H}$ [47, Equation 8.1]. Fix an operator $A \in \mathscr{S}(\mathcal{H})$ and let J be an involution that commutes with the selfadjoint extensions of A (there is always such involution because of [40, Proposition 2.3]). Consider a function $\xi_A : \mathbb{C} \to \mathcal{H}$ such that

(P1) $\xi_A(z)$ is zero-free ($\xi_A(z) \neq 0$ for all $z \in \mathbb{C}$) and entire,

(P2) $\xi_A(z) \in \ker(A^* - zI)$ for all $z \in \mathbb{C}$, and

(P3) $J\xi_A(z) = \xi_A(\bar{z})$ for every $z \in \mathbb{C}$.

Since $A \in \mathscr{S}(\mathcal{H})$, one has $\dim \ker(A^* - zI) = 1$ for all $z \in \mathbb{C}$. Using this fact, one can prove the following assertion (see [40, Proposition 2.12 and Remark 2.13]).

Lemma 4.3. *If* $\xi_A^{(1)} : \mathbb{C} \to \mathcal{H}$ *and* $\xi_A^{(2)} : \mathbb{C} \to \mathcal{H}$ *are two functions satisfying* (P1),(P2), *and* (P3), *then there exists a zero-free real entire function* $g(z)$ *such that* $\xi_A^{(1)}(z) = g(z)\xi_A^{(2)}(z)$.

There is a standard way of constructing a function $\xi_A(z)$ with the desired properties: Pick a canonical selfadjoint extension A_γ of $A \in \mathscr{S}(\mathcal{H})$ and let $h_\gamma(z)$ be a real entire function whose zero set (counting multiplicities) equals $\mathrm{spec}(A_\gamma)$; the existence of this function follows from classical theorems on entire functions, see [29, Chapter 7, §2]. Then, define

$$(18) \qquad \xi_A(z) := h_\gamma(z) V_{A_\gamma}(w, z)\psi_w,$$

where $w \in \mathbb{R} \setminus \mathrm{spec}(A_\gamma)$, $\psi_w \in \ker(A^* - wI) \setminus \{0\}$ and $V_{A_\gamma}(w, z)$ is given by (3). It can be shown that (18) obeys (P1), (P2) and (P3), the latter relative to a suitable involution [40, Propositions 2.3 and 2.11]. Note that Lemma 4.3 implies that (18) does not depend on the choice of the selfadjoint extension A_γ nor on w and, furthermore, every function $\xi_A : \mathbb{C} \to \mathcal{H}$ can be written in the form of (18).

Fix $A \in \mathscr{S}(\mathcal{H})$ and any function $\xi_A : \mathbb{C} \to \mathcal{H}$ satisfying (P1), (P2), and (P3). Then define the map

$$(19) \qquad (\Phi_A \varphi)(z) := \langle \xi_A(\bar{z}), \varphi \rangle, \qquad \varphi \in \mathcal{H}.$$

Due to (P1), Φ_A linearly maps $\mathcal{H}$ onto a linear manifold $\Phi_A \mathcal{H}$ of entire functions. By [45, §1], the complete nonselfadjointness condition

$$\bigcap_{z \in \mathbb{C} \backslash \mathbb{R}} \mathrm{ran}(A - zI) = \{0\}$$

(see [14, Chapter 1, Theorem 2.1]) implies that Φ_A is injective [42]. Clearly, the linear space $\Phi_A \mathcal{H}$ is turned into a Hilbert space by defining

$$(20) \qquad \langle \Phi_A \eta, \Phi_A \varphi \rangle := \langle \eta, \varphi \rangle.$$

The resulting Hilbert space is a de Branges space which is henceforth denoted by $\mathcal{B}_A$ [40, Proposition 2.14].

Remark 4.4. If $A \in \mathscr{S}(\mathcal{H})$, then $\Phi_A A \Phi_A^{-1}$ is the multiplication operator S in $\mathcal{B}_A$. For any (canonical) selfadjoint extension A_γ of A, $\Phi_A A_\gamma \Phi_A^{-1}$ is a (canonical) selfadjoint extension of the multiplication operator in $\mathcal{B}_A$. Moreover, if κ is a spectral gauge of A and $m(z) = (\Phi_A \kappa)(z)$, then (see Definition 3.1)

$$\mathcal{V}_\kappa(A) = \mathcal{V}_m(S) \quad \text{and} \quad \mathcal{V}_\kappa^{\mathrm{ext}}(A) = \mathcal{V}_m^{\mathrm{ext}}(S).$$

Note that, if F is a generalized spectral family of A, then $\Phi_A F \Phi_A^{-1}$ is a generalized spectral family of S.

The reproducing kernel in the de Branges space $\mathcal{B}_A$ is

$$k(z, w) = \langle \xi_A(\bar{z}), \xi_A(\bar{w}) \rangle.$$

Note that $k(z, w)$ is antientire with respect to its second argument.

Remark 4.5. It is not difficult to verify that $k(\cdot, w) \in \ker(S^* - \bar{w}I)$ for all $w \in \mathbb{C}$ and moreover

$$k(z, w) = h(\bar{w}) V_{S_\gamma}(\bar{v}, \bar{w}) k(z, v),$$

where S_γ is a canonical selfadjoint extension of S given in Definition 4.2 and $h(w)$ is a real entire function having zeros at $\text{spec}(S_\gamma)$ (see (6) and (18)).

Lemma 4.6. *If κ is a spectral gauge of $A \in \mathcal{S}(\mathcal{H})$, then the corresponding element m in $\mathcal{B}_A$ (given by Remark 4.4) has no zeros in $\mathbb{R}$. Moreover, if the function m in the de Branges space $\mathcal{B}$ is such that $m(x) \neq 0$ for all $x \in \mathbb{R}$, then m is a spectral gauge of the operator of multiplication S in $\mathcal{B}$.*

Proof. By our definition of spectral gauge,

$$\langle \xi(\bar{z}), \kappa \rangle \neq 0$$

for any $z \in \mathbb{R}$. The second part of the statement follows from noticing that $k(\cdot, w) \in \ker(S^* - \bar{w}I)$ for all $w \in \mathbb{C}$. $\qquad \square$

The following definition coincides with [35, Definition 6.1]. We came to it motivated by a generalization of the moment problem discussed in [27] (see also [45]). Several results in [8] pertain to this definition.

Definition 4.7 (generalized moment problem). Given a de Branges space $\mathcal{B}$, find a Borel measure ρ such that

$$\langle g, f \rangle_{\mathcal{B}} = \int_{\mathbb{R}} f(x) \overline{g(x)} \, d\rho(x)$$

for every $f, g \in \mathcal{B}$.

As (17) shows, the Lebesgue measure multiplied by $1/|e(\cdot)|^2$ is always a solution to the generalized moment problem for $\mathcal{B}(e)$. Below, it will be established that there are other solutions to this generalized moment problem.

Theorem 4.8. *Let S be the operator of multiplication by the independent variable in a de Branges space $\mathcal{B}$ and m be a function in $\mathcal{B}$ not vanishing in $\mathbb{R}$. If ρ is a solution to the generalized moment problem for $\mathcal{B}$, then*

$$\sigma(\partial) := \int_{\partial} |m(x)|^2 \, d\rho(x), \quad \partial \in \mathfrak{X},$$

belongs to $\mathcal{V}_m(S)$.

Proof. Fix $t \in \mathbb{R}$ and define

$$\Upsilon_t(f, g) := \int_{-\infty}^{t} f(x) \overline{g(x)} \, d\rho(x)$$

which is a sesquilinear form in $\mathcal{B} \times \mathcal{B}$. This form is bounded due to the inequality

$$\Upsilon_t(f, f) \leq \int_{\mathbb{R}} |f(x)|^2 d\rho(x) = \|f\|^2$$

along with the polarization identity. By [6, Chapter 2, §4, Theorem 6] and well-known results on sesquilinear forms [33, §II.2], there is a bounded operator $F(t)$ defined on the whole space $\mathcal{B}$ such that

$$\langle g, F(t)f \rangle = \Upsilon_t(f, g).$$

It turns out that $F(t)$ corresponds to a generalized spectral family in the sense of Proposition 2.6. Indeed, (i)–(iii) are verified directly from the definition. As regards (iv), one verifies

$$\int_{\mathbb{R}} t \, d\langle g, F(t)f \rangle = \int_{\mathbb{R}} t f(t) \overline{g(t)} d\rho(t) = \langle g, Sf \rangle_{\mathcal{B}}$$

and

$$\int_{\mathbb{R}} t^2 d\langle f, F(t)f \rangle = \int_{\mathbb{R}} t^2 f(t) \overline{f(t)} d\rho(t) = \langle Sf, Sf \rangle_{\mathcal{B}}.$$

Thus, for any real Borel set ∂,

$$\mathcal{V}_m(S) \ni \langle m, F(\partial)m \rangle = \int_{\partial} |m(x)|^2 \, d\rho(x)$$

since m is a spectral gauge of S by Lemma 4.6. □

Lemma 4.9. *Let $m \in \mathcal{B}$ be such that $m(x) \neq 0$ when $x \in \mathbb{R}$ and fix $a, b \in \mathbb{R}$ such that $a < b$. If F is a generalized spectral family of S, then*

$$(F([a, b])f)(z) = \int_a^b \frac{f(t)}{m(t)} \, dF(t)m(z)$$

for any $f \in \mathcal{B}$.

Proof. According to Remark 4.5, since $m(z)$ is a spectral gauge of S, the right-hand side of (12) in this case is

$$\int_a^b \frac{\langle k(\cdot, t), f(\cdot) \rangle_{\mathcal{B}}}{\langle k(\cdot, t), m(\cdot) \rangle_{\mathcal{B}}} dF(t)m(z).$$

The assertion then follows from Proposition 2.16. □

Theorem 4.10. *Let S be the operator of multiplication by the independent variable in a de Branges space $\mathcal{B}$ and m be a function in $\mathcal{B}$ not vanishing in $\mathbb{R}$. If $\sigma \in \mathcal{V}_m(S)$, then*

$$\rho(\partial) := \int_{\partial} \frac{d\sigma(x)}{|m(x)|^2}, \quad \partial \in \mathfrak{X},$$

is a solution to the generalized moment problem for $\mathcal{B}$.

Proof. By the hypothesis and Lemma 4.9,

$$\sigma(\partial) = \langle m, F(\partial)m \rangle_{\mathcal{B}}$$

for any real Borel set ∂. It follows from Lemma 4.9 that

$$\langle g, F[a, b]f \rangle_{\mathcal{B}} = \int_a^b \frac{f(t)}{m(t)} d\langle g, F(t)m \rangle_{\mathcal{B}}.$$

Also

$$\langle F(t)g, m \rangle_{\mathcal{B}} = \langle F(a)g, m \rangle_{\mathcal{B}} + \int_a^t \frac{\overline{g(x)}}{m(x)} d\langle F(x)m, m \rangle_{\mathcal{B}}$$

Putting the last two equalities together, one obtains

$$\langle g, F[a, b]f \rangle_{\mathcal{B}} = \int_a^b \frac{f(t)\overline{g(t)}}{|m(t)|^2} d\sigma(t) = \int_a^b f(t)\overline{g(t)} d\rho(t)$$

To finish the proof, take the limit when $a \to -\infty$ and $b \to +\infty$.   □

Lemma 4.11. *Let S be the operator of multiplication by the independent variable in a de Branges space $\mathcal{B}$ and m be a function in $\mathcal{B}$ not vanishing in $\mathbb{R}$. If S_γ is a canonical selfadjoint extension of S and*

$$\sigma(\partial) := \langle m, E_\gamma(\partial)m \rangle, \quad \partial \in \mathfrak{X},$$

where E_γ is the spectral family of S_γ, then

$$(\mathfrak{f}(S_\gamma)m)(\lambda) = \mathfrak{f}(\lambda)m(\lambda)$$

for any $\mathfrak{f}$ in $L_2(\mathbb{R}, \sigma)$ and $\lambda \in \mathrm{spec}(S_\gamma)$.

Proof. Using the spectral theorem,

$$(\mathfrak{f}(S_\gamma)m)(z) = \left(\int_{\mathbb{R}} \mathfrak{f}(t)dE_\gamma(t)m \right)(z) = \sum_{\lambda \in \mathrm{spec}(S_\gamma)} \mathfrak{f}(\lambda)k(z, \lambda)\frac{\langle k(\cdot, \lambda), m \rangle_{\mathcal{B}}}{k(\lambda, \lambda)}.$$

From the last expression the result follows due to the properties of the reproducing kernel.   □

Theorem 4.12. *Let $\mathcal{B}$, S, and m be as in the previous theorem. Assume that ρ is a solution to the generalized moment problem and σ is related to it as in Theorem 4.10. A necessary and sufficient condition for the measure σ to be in $\mathcal{V}_m^{\mathrm{ext}}(S)$ is that the map Ψ between the spaces $\mathcal{B}$ and $L_2(\mathbb{R}, \rho)$ given by*

$$\Psi f = f \restriction_{\mathbb{R}}$$

is unitary (that is, linear, surjective and norm-preserving).

Proof. Since ρ is a solution to the generalized moment problem, $\Psi\mathcal{B}$ is contained in $L_2(\mathbb{R}, \rho)$ and Ψ is isometric. Assume that Ψ is unitary (by what has just been stated, this means that one only assumes $\{f \restriction_{\mathbb{R}}: f \in \mathcal{B}\} = L_2(\mathbb{R}, \rho)$). Let A^ρ be the operator of multiplication by the independent variable in $L_2(\mathbb{R}, \rho)$ defined in the maximal domain. Since A^ρ is selfadjoint, the same holds for $\Psi^{-1}A^\rho\Psi$ due to the fact that Ψ is unitary. If $f \in \mathrm{dom}(S)$, then $\Psi f \in \mathrm{dom}(A^\rho)$. Thus, $\Psi^{-1}A^\rho\Psi$ is a canonical selfadjoint extension of S. Denote $S_\gamma := \Psi^{-1}A^\rho\Psi$ and let E_γ be the spectral family of S_γ. For any Borel set $\partial \subset \mathbb{R}$, define $\eta(\partial) := \langle m, E_\gamma(\partial)m \rangle$. By the canonical map (see [2, §69]),

$$L_2(\mathbb{R}, \eta) \ni \mathfrak{f} \mapsto \mathfrak{f}(S_\gamma)m \in \mathcal{B},$$

and for any function f in $\mathcal{B}$ there is a function $\mathfrak{f}$ in $L_2(\mathbb{R}, \eta)$ such that $f = \mathfrak{f}(S_\gamma)m$. Thus

$$\int_{\mathbb{R}} f(t)\overline{g(t)}\, d\rho(t) = \int_{\mathbb{R}} \mathfrak{f}(t)\overline{\mathfrak{g}(t)}\, d\eta(t),$$

where f and g are the images of $\mathfrak{f}$ and $\mathfrak{g}$ under the canonical map. On the basis of Lemma 4.11, $f(t) = \mathfrak{f}(t)m(t)$ and $g(t) = \mathfrak{g}(t)m(t)$ with the equalities in the $L_2(\mathbb{R}, \eta)$ sense. Therefore

$$\int_{\mathbb{R}} f(t)\overline{g(t)}\, d\rho(t) = \int_{\mathbb{R}} f(t)\overline{g(t)}\frac{d\eta(t)}{|m(t)|^2}.$$

Since this equality holds for any element in $L_2(\mathbb{R}, \rho)$, one concludes that $\eta(\partial) = \sigma(\partial)$ for any real Borel set ∂ and hence σ is extremal, that is, $\sigma \in \mathcal{V}_m^{\text{ext}}(S)$.

Let us prove the other direction. If $\sigma \in \mathcal{V}_m^{\text{ext}}(S)$, then σ is the spectral measure of a canonical selfadjoint extension of S which is known to be simple. By the canonical map (see [2, §69]), $\mathcal{B}$ is unitarily equivalent to $L_2(\mathbb{R}, \sigma)$. In turn, since m is zero-free on the real axis, $L_2(\mathbb{R}, \sigma)$ is unitarily equivalent to $L_2(\mathbb{R}, \rho)$. On the other hand, the Hilbert spaces $\mathcal{B}$ and $\Psi\mathcal{B} \subset L_2(\mathbb{R}, \rho)$ are unitarily equivalent due to the fact that ρ is a solution to the moment problem. Therefore, $\Psi\mathcal{B}$ cannot be properly contained in $L_2(\mathbb{R}, \rho)$. $\square$

Definition 4.13. A solution ρ to the generalized moment problem for $\mathcal{B}$ is said to be extremal if the map $\Psi : \mathcal{B} \to L_2(\mathbb{R}, \rho)$ given in Theorem 4.12 is unitary.

Unlike the extremal measures (see Definition 3.1), the extremal solutions might not be finite measures. However, the extremal solutions of the generalized moment problem for $\mathcal{B}$ can be finite measures if, for instance, $1 \in \mathcal{B}$ (see Section 6.1).

Let $\delta_x : \mathfrak{X} \to [0, 1]$ be the measure defined by the rule

$$\delta_x(\partial) := \begin{cases} 1 & x \in \partial, \\ 0 & x \notin \partial. \end{cases}$$

Theorem 4.14. *Let $\mathcal{B}$ be a de Branges space, $k(\cdot, w)$ its reproducing kernel, and S_γ a canonical selfadjoint extension of the operator of multiplication S. The measure*

$$\rho_\gamma := \sum_{\lambda \in \text{spec}(S_\gamma)} \frac{\delta_\lambda}{k(\lambda, \lambda)} \tag{21}$$

is an extremal solution to the generalized moment problem for $\mathcal{B}$. Reciprocally, every extremal solution is of the form (21).

Proof. It follows from Remark 4.5 and the equality

$$\ker(S_\gamma - \lambda I) = \ker(S^* - \lambda I)$$

that $\{k(\cdot, \lambda)/\|k(\cdot, \lambda)\|\}_{\lambda \in \text{spec}(S_\gamma)}$ is an orthonormal basis in $\mathcal{B}$. Therefore, a function f is in $\mathcal{B}$ if and only if

$$\sum_{\lambda \in \text{spec}(S_\gamma)} \frac{|f(\lambda)|^2}{k(\lambda, \lambda)} < +\infty \tag{22}$$

and one has the following interpolation formula:

$$f(z) = \sum_{\lambda \in \text{spec}(S_\gamma)} \frac{k(z, \lambda)}{k(\lambda, \lambda)} f(\lambda) \tag{23}$$

holds, where the convergence is in the sense of Hilbert space, hence uniform in compact subsets of $\mathbb{C}$. Thus, if ρ_γ is given by (21), then every f in $L_2(\mathbb{R}, \rho_\gamma)$ satisfies (22) and, through the interpolation formula (23), it is mapped into a function in $\mathcal{B}$ whose restriction to the real axis is the function f. The identity

$$\|f\|_{\mathcal{B}}^2 = \sum_{\lambda \in \text{spec}(S_\gamma)} \frac{|f(\lambda)|^2}{k(\lambda, \lambda)} = \|f\|_{L_2(\mathbb{R}, \rho_\gamma)}^2$$

implies that Ψ is norm-preserving.

Conversely, if ρ is an extremal solution, then $\text{supp}\, \rho$ coincides with $\text{spec}(S_\gamma)$ for some γ as a consequence of Theorem 4.12. Therefore, if one fixes $\lambda \in \text{spec}(S_\gamma)$ and considers any f in $L_2(\mathbb{R}, \rho)$ whose zero set is $\text{spec}(S_\gamma) \setminus \{\lambda\}$, then

$$\frac{|f(\lambda)|^2}{k(\lambda, \lambda)} = \|f\|_{\mathcal{B}}^2 = \|f\|_{L_2(\mathbb{R}, \rho)}^2 = c|f(\lambda)|^2.$$

This shows that the weight c of the measure ρ at λ is equal to $1/k(\lambda, \lambda)$. $\qquad\square$

Remark 4.15. Assume that $A \in \mathcal{G}(\mathcal{H})$ and take the measure ρ_γ as defined in Theorem 4.14 for the corresponding de Branges space $\mathcal{B}_A$. The multiplication by the independent variable in $L_2(\mathbb{R}, \rho_\gamma)$ is unitarily equivalent to a canonical selfadjoint extension A_γ of A. Note that one can write

$$\rho_\gamma = \sum_{\lambda \in \text{spec}(S_\gamma)} \frac{\delta_\lambda}{\|\xi_A(\lambda)\|^2}.$$

It is worth remarking that in the Sturm–Liouville theory (when A is associated with a regular Sturm–Liouville difference expression) the measure ρ_γ given above is usually defined via the Weyl function and is called the spectral measure of the selfadjoint operator A_γ (see details in Section 6.2).

5. Point mass perturbations of measures

Lemma 5.1. *Let $\mathcal{B}$ be a de Branges space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{B}}$. Given $a > 0$ and $\lambda \in \mathbb{R}$, define*

$$\langle g, f \rangle_\sim := \langle g, f \rangle_{\mathcal{B}} + a \overline{g(\lambda)} f(\lambda), \quad f, g \in \mathcal{B}. \tag{24}$$

Then the linear manifold $\mathcal{B}$ equipped with the inner product $\langle \cdot, \cdot \rangle_\sim$ is a de Branges space.

Proof. $\mathcal{B}$ is closed with respect to $\langle \cdot, \cdot \rangle_\sim$. Indeed, let $\{f_n\}_0^\infty \subset \mathcal{B}$ be a $\sim$-Cauchy sequence. Since $\|\cdot\|_\sim \geq \|\cdot\|_{\mathcal{B}}$, $\{f_n\}_0^\infty$ is also $\mathcal{B}$-Cauchy. Then, there exists $g \in \mathcal{B}$ such that $\|f_n - g\|_{\mathcal{B}} \to 0$. Since $|f_n(\lambda) - g(\lambda)|^2 \leq k(\lambda, \lambda)\|f_n - g\|_{\mathcal{B}}^2$, one obtains

$$\|f_n - g\|_\sim^2 \leq \|f_n - g\|_{\mathcal{B}}^2 + ak(\lambda, \lambda)\|f_n - g\|_{\mathcal{B}}^2,$$

implying the assertion.

Now we prove (A1)–(A3) of Definition 4.1. Since

$$|f(w) - g(w)|^2 = |\langle k(\cdot, w), f - g \rangle|^2 \leq k(w, w)\|f - g\|_\mathcal{B}^2 \leq k(w, w)\|f - g\|_\sim^2,$$

it follows that point evaluation is continuous with respect to the $\sim$-norm.

From the equality

$$\|f^\#\|_\sim^2 = \|f^\#\|_\mathcal{B}^2 + a|f^\#(\lambda)|^2 = \|f\|_\mathcal{B}^2 + a|f(\lambda)|^2 = \|f\|_\sim^2,$$

it follows that the mapping $f \mapsto f^\#$ (see (A3) in Definition 4.1) is a $\sim$-isometry in $\mathcal{B}$.

Finally, suppose $w \in \mathbb{C} \setminus \mathbb{R}$ is a zero of $f \in \mathcal{B}$ and define $g(z) := (z - \bar{w})(z - w)^{-1} f(z)$. Then, $g \in \mathcal{B}$ and

$$\|g\|_\sim^2 = \|g\|_\mathcal{B}^2 + a\left|\frac{\lambda - \bar{w}}{\lambda - w}\right|^2 |f(x)|^2 = \|f\|_\mathcal{B}^2 + a|f(x)|^2 = \|f\|_\sim^2. \qquad \square$$

Lemma 5.1 has the following corollary.

Corollary 5.2. *Let $\mathcal{B}$ be a de Branges space. If ρ is a solution to the generalized moment problem for $\mathcal{B}$, then $\rho + a\delta_\lambda$ ($a > 0$, $\lambda \notin \operatorname{supp} \rho$) is a solution to the generalized moment problem for a de Branges space $\mathcal{B}_\sim$ having the same elements as $\mathcal{B}$ but with inner product given by* (24).

Proof. The statement follows from the equality

$$\int_\mathbb{R} \overline{g(x)} f(x) \, d(\rho + a\delta_\lambda) = \langle g, f \rangle_\mathcal{B} + a\overline{g(\lambda)} f(\lambda). \qquad \square$$

Theorem 5.3. *Let S be the operator of multiplication by the independent variable in a de Branges space $\mathcal{B}$ and m be a function in $\mathcal{B}$ not vanishing in $\mathbb{R}$. If $\sigma \in \mathcal{V}_m(S)$, then*

$$\sigma + a|m(\lambda)|^2\delta_\lambda, \quad a > 0, \ \lambda \notin \operatorname{supp}(\sigma),$$

is in $\mathcal{V}_m(\tilde{S}) \setminus \mathcal{V}_m^{\mathrm{ext}}(\tilde{S})$, where $\tilde{S}$ is the multiplication operator in the de Branges space $\mathcal{B}_\sim$ given in Corollary 5.2.

Proof. By hypothesis, ρ given in Theorem 4.10 is a solution to the generalized moment problem for $\mathcal{B}$. By Corollary 5.2, $\rho + a\delta_\lambda$ is a solution to the generalized moment problem for $\mathcal{B}_\sim$ and Theorem 4.8 implies that

$$\sigma + a|m(\lambda)|^2\delta_\lambda \in \mathcal{V}_m(\tilde{S}).$$

On the other hand, Corollary 3.3 yields a measure $\sigma_\lambda \in \mathcal{V}_m^{\mathrm{ext}}(S)$ such that $\sigma_\lambda\{\lambda\} > 0$. The function ρ_λ, related to σ_λ as in Theorem 4.10, is a solution to the generalized moment problem for $\mathcal{B}$. Recurring again to Corollary 5.2, one concludes that $\rho_\lambda + a\delta_\lambda$ is a solution to the generalized moment problem for $\mathcal{B}_\sim$. Thus, in view of Theorem 4.8,

$$\sigma_\lambda + a|m(\lambda)|^2\delta_\lambda \in \mathcal{V}_m(\tilde{S}).$$

Since $\lambda \in \operatorname{supp}(\sigma_\lambda) \setminus \operatorname{supp}(\sigma)$, one has

$$(\sigma_\lambda + a|m(\lambda)|^2\delta_\lambda)\{\lambda\} > (\sigma + a|m(\lambda)|^2\delta_\lambda)\{\lambda\} > 0.$$

Assume that $\sigma + a|m(\lambda)|^2\delta_\lambda$ is extremal. If $\sigma_\lambda + a|m(\lambda)|^2\delta_\lambda$ is extremal, then a contradiction arises from Corollary 3.3. If $\sigma_\lambda + a|m(\lambda)|^2\delta_\lambda$ is not extremal, then, using Theorem 3.6, one obtains an extremal

measure giving weight to λ so again we have a contradiction by Corollary 3.3. Therefore $\sigma + a|m(\lambda)|^2 \delta_\lambda$ is not in $\mathcal{V}_m^{\text{ext}}(\tilde{S})$. $\qquad\square$

Theorem 5.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be de Branges spaces such that they are set-wise equal. If ρ is a solution to the generalized moment problem for $\mathcal{B}$, then $\rho + a\delta_\lambda$ ($a > 0$, $\lambda \notin \operatorname{supp} \rho$) is not an extremal solution to the generalized moment problem for $\mathcal{B}'$.*

Remark 5.5. Before proving the assertion, there are two points to comment. First, the measure $\rho + a\delta_\lambda$ is not necessarily a solution of the generalized moment problem for $\mathcal{B}'$. Second, the particular case when in the hypothesis ρ is an extremal solution has interesting applications (see Corollary 5.7 and Section 6).

Proof. Assume that $\rho + a\delta_\lambda$ is an extremal solution to the generalized moment problem for $\mathcal{B}'$. Thus,

$$\|f\|_{\mathcal{B}'}^2 = \int_{\mathbb{R}} |f(x)|^2 \, d\rho + a|f(\lambda)|^2$$

By hypothesis,

$$\|f\|_{\mathcal{B}}^2 = \int_{\mathbb{R}} |f(x)|^2 \, d\rho.$$

Therefore the inner products of $\mathcal{B}$ and $\mathcal{B}'$ are related as in (24). Now, since ρ and $\rho + a\delta_\lambda$ generate via Theorem 4.8 spectral measures of the operators of multiplication in $\mathcal{B}$ and $\mathcal{B}'$, one obtains a contradiction from Theorem 5.3. $\qquad\square$

Remark 5.6. There is an alternative proof of Theorem 5.4. Indeed, assume that $\rho + a\delta_\lambda$ is an extremal solution of the moment problem for $\mathcal{B}'$ and consider a function $g \in L_2(\mathbb{R}, \rho + a\delta_\lambda)$ whose zero set is $\operatorname{supp} \rho$. Thus, $\|g\|_{L_2(\mathbb{R}, \rho+a\delta_\lambda)} \neq 0$ and, since Ψ is surjective and norm preserving, the function $\Psi^{-1}g$ is a nonzero element of $\mathcal{B}'$ and therefore a nonzero element of $\mathcal{B}$. But we get a contradiction since the norm of the restriction to the real line of $\Psi^{-1}g$ has zero norm in $L_2(\mathbb{R}, \rho)$.

Corollary 5.7. *Let ρ be a solution to the generalized moment problem for a de Branges space $\mathcal{B}$. If $\mathcal{A} \subset \Psi\mathcal{B}$ is dense in $L_2(\mathbb{R}, \rho)$, then $\mathcal{A}$ is not dense in $L_2(\mathbb{R}, \rho + a\delta_\lambda)$, where $a > 0$ and $x \notin \operatorname{supp}(\rho)$.*

Proof. The hypothesis implies that ρ is an extremal solution, i.e., $\Psi\mathcal{B} = L_2(\mathbb{R}, \rho)$. As in Corollary 5.2, let $\mathcal{B}_\sim$ be the de Branges space having the same elements of $\mathcal{B}$ with the inner product given by (24). Then $\Psi\mathcal{B}_\sim$, which contains $\mathcal{A}$, is not dense in $L_2(\mathbb{R}, \rho + a\delta_\lambda)$ as a consequence of Theorem 5.4. $\qquad\square$

Remark 5.8. Note that a function is square-integrable with respect to ρ if and only if it is square-integrable with respect to $\rho + a\delta_\lambda$. Nevertheless, there are functions in the equivalence class of zero in $L_2(\mathbb{R}, \rho)$ which have nonzero norm in $L_2(\mathbb{R}, \rho + a\delta_\lambda)$.

6. Examples

6.1. The classical moment problem. According to [1, Theorem 2.1.1], for a given real sequence $\{s_k\}_{k=0}^\infty$, there exists a Borel measure ρ such that

$$s_k = \int_{\mathbb{R}} t^k \, d\rho(t) \quad \text{for } k = 0, 1, \dots$$

if and only if, for all $k \in \mathbb{N}$,

$$\text{(25)} \qquad \det \begin{pmatrix} s_0 & \cdots & s_k \\ \vdots & \vdots & \vdots \\ s_k & \cdots & s_{2k} \end{pmatrix} > 0.$$

The measure is said to be a solution to the moment problem given by the sequence $\{s_k\}_{k=0}^{\infty}$. We refer to all sequences $\{s_k\}_{k=0}^{\infty}$ satisfying (25) and normalized so that $s_0 = 1$ as sequences of moments. To any sequence of moments there corresponds one and only one Jacobi matrix

$$\text{(26)} \qquad \begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \\ 0 & b_2 & q_3 & b_3 & \ddots \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $\{q_k\}_{k=1}^{\infty}$ is a sequence of real numbers and $\{b_k\}_{k=1}^{\infty}$ is a sequence of positive numbers (see [1, paragraph after Equation 1.8] and [43, p. 93]).

Given a separable Hilbert space $\mathcal{H}$ and an orthonormal basis $\{\eta_k\}_{k=1}^{\infty}$, with any matrix (26), one can uniquely associate a closed symmetric operator [2, §47]. This operator, denoted by A and called minimal Jacobi operator, has either deficiency indices $n_+(A) = n_-(A) = 0$ or $n_+(A) = n_-(A) = 1$ [5, Chapter 7, Theorem 1.1]. When the Jacobi operator has deficiency indices $(0, 0)$ the matrix (26) is said to be in the limit point case, otherwise the matrix (26) is said to be in the limit circle case. Due to the bijection between Jacobi matrices and sequences of moments, the limit point and limit circle dichotomy corresponds to the determinate and indeterminate dichotomy for sequences of moments.

It is established in [1, Chapter 4] that

$$\text{(27)} \qquad s_k = \langle \eta_1, A^k \eta_1 \rangle \quad \text{for } k = 0, 1, \ldots$$

This shows that if A has deficiency indices $n_+(A) = n_-(A) = 1$, then the spectral measures of canonical selfadjoint extensions are solutions of the moment problem, i.e., an indeterminate sequence of moments admits various solutions. According to [1, Corollary 2.2.4], when the deficiency indices $n_+(A)$ and $n_-(A)$ vanish, there are only one solution to the moment problem. This justifies the terminology.

A Jacobi operator A with deficiency indices $n_+(A) = n_-(A) = 1$ is in $\mathscr{S}(\mathcal{H})$. Indeed, if one assumes that λ is in the spectral kernel of A, then $\lambda \in \text{spec}(A_\gamma)$ for any selfadjoint extension A_γ of A since the spectral kernel does not decrease under extensions. But the spectra of canonical selfadjoint extensions of A are disjoint (see the proof of [1, Theorem 4.2.4]. We assume henceforth in this example that the Jacobi operator A is not selfadjoint.

The function defined in (18) is given in this case by $\xi_A(z) = \sum_{k=1}^{\infty} P_{k-1}(z)\eta_k$, where P_k is the k-th orthogonal polynomial of the first kind associated with (26) (see [1, Chapter 1, §2; Chapter 4, §1] and [5, Chapter 7, §2]). Thus we have the unitary map $\Phi_A : \mathcal{H} \to \mathcal{B}_A$ given by (19). Note first that, since the finite linear combinations of the basis $\{\eta_k\}_{k=1}^{\infty}$ are dense in $\mathcal{H}$, the polynomials are dense in $\mathcal{B}_A$. This

is a peculiarity of de Branges spaces generated by Jacobi operators. Also,

$$(\Phi_A A \eta_1)(z) = \langle \xi_A(\bar{z}), A \eta_1 \rangle = \langle A^* \xi_A(\bar{z}), \eta_1 \rangle = \langle \bar{z} \xi_A(\bar{z}), \eta_1 \rangle = z$$

since $\langle \xi_A(\bar{z}), \eta_1 \rangle = 1$. Thus

(28)
$$(\Phi_A A^k \eta_1)(z) = z^k \quad \text{for all } k \in \mathbb{N} \cup \{0\} .$$

Theorem 6.1. *The measure ρ is a solution to an indeterminate moment problem if and only if ρ is a solution to the generalized moment problem for $\mathcal{B}_A$.*

Proof. Assume that ρ is a solution to the generalized moment problem for $\mathcal{B}_A$. Then, in view of (27) and (28), one has

$$s_k = \langle 1, z^k \rangle_{\mathcal{B}_A} = \int_{\mathbb{R}} t^k \, d\rho(t) \quad \text{for } k = 0, 1, \dots$$

which means that ρ is a solution to the moment problem given by $\{s_k\}_{k=0}^\infty$.

Now, suppose that ρ is a solution to the moment problem given by $\{s_k\}_{k=0}^\infty$. Hence, by (27), one has

$$\int_{\mathbb{R}} t^k \, d\rho(t) = \langle \eta_1, A^k \eta_1 \rangle \quad \text{for } k = 0, 1, \dots,$$

where A is the Jacobi operator associated with the sequence $\{s_k\}_{k=0}^\infty$. One then verifies that if $R(t)$ is a polynomial, then

$$\int_{\mathbb{R}} |R(t)|^2 \, d\rho(t) = \langle R(A)\eta_1, R(A)\eta_1 \rangle = \|R\|_{\mathcal{B}_A},$$

where (28) is used to obtain the second equality.

For completing the proof, one uses the fact that the polynomials are dense in $\mathcal{B}_A$ and the polarization identity. $\qquad\square$

In the context of the classical moment problem, Theorem 5.3 corresponds to [36, Proposition 4.1(a)], which says that if ρ is an extremal solution to an indeterminate moment problem, then $\rho + a\delta_\lambda$ ($a > 0$, $\lambda \notin \operatorname{supp} \rho$) is not extremal although is a solution to a moment problem. Thus, according to Corollary 5.7, the density of the polynomials in $L_2(\mathbb{R}, \rho)$ no longer holds in $L_2(\mathbb{R}, \rho + a\delta_\lambda)$. Note that the sequences of moments for which ρ and $\rho + a\delta_\lambda$ are solutions do not coincide, therefore the corresponding Jacobi operators differ from one another, as well as the related de Branges spaces. However, these de Branges spaces are set-wise equal, which is actually a consequence of the fact that the set of polynomials are dense in these two spaces and the norms are equivalent (see the proof of Lemma 5.1).

We conclude this example by pointing out that it was probably first mentioned in [13, Remark 4.5 (iii)] that the set of points obtained by adding a point to the spectrum of a selfadjoint extension of a discrete Schrödinger operator in the limit circle case is no longer the spectrum of a selfadjoint extension of a discrete Schrödinger operator in the limit circle case (see [12, §7, Example 2] and [26, Chapter 2, §7]).

6.2. Schrödinger operators with measures. The following example is based on [4] (see also [28]). Let us consider the differential expression given informally by

$$\tau_\mu := -\frac{d^2}{dx^2} + \mu, \quad x \in [0, b],$$

where μ is a signed Borel measure on $[0, b]$. Properly, for a function φ in $AC[0, b]$ (the space of absolutely continuous functions on $[0, b]$), we define

$$(29) \qquad \varphi^{[1]}(x) := \varphi'(x) - \int_{[0,x]} \varphi(t)\, d\mu(t),$$

then we define

$$(30) \qquad \mathrm{dom}(\tau_\mu) := \{\varphi \in AC[0, b] : \varphi^{[1]} \in AC[0, b]\}, \quad \tau_\mu \varphi := -(\varphi^{[1]})'.$$

The derivative of an element $\varphi \in \mathrm{dom}(\tau_\mu)$ has a (unique) representative for which (29) holds for every $x \in [0, b]$; in what follows φ' will denote this particular representative. From [4, Theorem 2.4] it follows that φ' may have discontinuities. Indeed,

$$(31) \qquad \varphi'(x) - \varphi'(x-) = \varphi(x)\mu(\{x\}), \quad x \in (0, b].$$

Let $\xi(z, x)$ be the solution to the eigenvalue equation $\tau_\mu \varphi = z\varphi$, $z \in \mathbb{C}$, in the sense given by (30), that satisfies the initial conditions $\xi(z, 0) = 1$, $\xi^{[1]}(z, 0) = 0$. By [4, Theorem 2.3], this solution exists and is a real entire function of z for every fixed $x \in [0, b]$.

Let A denote the symmetric operator in $L_2(0, b)$ given by

$$(32) \quad \mathrm{dom}(A) := \{\varphi \in \mathrm{dom}(\tau_\mu) : \tau_\mu \varphi \in L_2(0, b),\ \varphi^{[1]}(0) = \varphi'(b-) = \varphi(b) = 0\}, \quad A\varphi := -(\varphi^{[1]})'.$$

The boundary condition $\varphi^{[1]}(0) = 0$ is just the usual one $\varphi'(0) + h\varphi(0) = 0$ with $h = -\mu(\{0\})$. On the other hand, the boundary condition at b is consequence of (31) plus the fact that A is the closure of the minimal operator.

Standard arguments yields that A has deficiency indices $(1, 1)$ and $\xi(z, \cdot) \in \ker(A^* - zI)$ for all $z \in \mathbb{C}$. Since Green's identity holds true for this kind of operators [4, Theorem 2.2], its canonical selfadjoint extensions are defined in the usual way,

$$\mathrm{dom}(A_\gamma) := \left\{ \begin{array}{l} \varphi \in \mathrm{dom}(\tau_\mu) : \tau_\mu \varphi \in L_2(0, b),\ \varphi^{[1]}(0) = 0, \\ \varphi(b)\cos\gamma + \varphi'(b)\sin\gamma = 0,\ \text{fixed } \gamma \in [0, \pi) \end{array} \right\}, \quad A_\gamma \varphi := -(\varphi^{[1]})'.$$

The de Branges space associated to A is given by

$$(33) \qquad \mathcal{B}_A := \left\{ f(z) = \int_0^b \xi(z, x)\varphi(x)\, dx : \varphi \in L_2(0, b) \right\}, \quad \|f\|_{\mathcal{B}_A} := \|\varphi\|_{L_2(0,b)}.$$

$\mathcal{B}_A$ is isometrically equal to $\mathcal{B}(e_b)$, where $e_b(z) = \xi(z, b) + i\xi'(z, b)$ is an Hermite–Biehler function as shown in [4, Proposition 4.1]. Thus, since A is unitarily equivalent to the multiplication operator in $\mathcal{B}_A$, $A \in \mathscr{S}(L_2(0, b))$.

The solutions of the generalized moment problem for $\mathcal{B}_A$ correspond to the generalized spectral measures of A. Let us consider here the extremal solutions of the generalized moment problem. According to Theorem 4.14 and Remark 4.15, any extremal solution is given by

$$(34) \qquad \rho_\gamma = \sum_{\lambda \in \mathrm{spec}(A_\gamma)} \frac{\delta_\lambda}{\|\xi(\cdot, \lambda)\|_{L_2(0,b)}^2},$$

where A_γ is a selfadjoint extension of A. As in the classical Sturm–Liouville theory, the Fourier transform introduced in (33) generates an unitary map between $L_2(0, b)$ and $L_2(\mathbb{R}, d\rho_\gamma)$ [4, §3].

Theorem 6.2. *Let A be the generalized Schrödinger operator defined by τ_μ as in (32). Let A_γ be an arbitrarily chosen selfadjoint extension of A and ρ_γ the corresponding spectral measure given in (34). If*

$$\tilde{\rho} = \rho_\gamma + s\delta_\lambda$$

with $\lambda \notin \mathrm{spec}(A_\gamma)$ and $s > 0$, then:

(a) *There is no signed Borel measure v in $[0,b]$ such that $\tilde{\rho}$ is the spectral measure corresponding to selfadjoint extensions of $\tilde{A}$ generated by τ_v.*

(b) *The set of functions*

$$(35) \qquad \left\{ f(y) := \int_0^b \cos(\sqrt{y}x)\varphi(x)\, dx : \varphi \in L_2(0, b) \right\}, \quad y \in \mathbb{R},$$

has no trivial orthogonal complement in $L_2(\mathbb{R}, \tilde{\rho})$.

Proof. According to [4, Theorem 4.4], the set of functions in the de Branges space $\mathcal{B}_A$ coincides with the linear set

$$\left\{ f(z) = \int_0^b \cos(\sqrt{z}x)\varphi(x)\, dx : \varphi \in L_2(0, b) \right\},$$

for any signed Borel measure supported in $[0, b]$. Then, assertion (a) is a direct consequence of Theorem 5.4 while (b) follows from Theorems 4.12 and 5.3. $\qquad\square$

Remark 6.3. We note that regular Schrödinger operators are included in Theorem 6.2 as they correspond to signed Borel measures μ that are absolutely continuous with respect to the Lebesgue measure, i.e.,

$$(36) \qquad d\mu(x) = q(x)\, dx \quad \text{with } q \in L_1(0, b).$$

For these cases, assertion (a) of Theorem 6.2 can be also shown using an elementary argument based on the fact that, the eigenvalues of a regular Schrödinger operator with Neumann boundary condition at the origin and any selfadjoint condition at b (Dirichlet case excluded) obey an asymptotic formula of the form

$$(37) \qquad \lambda_n = cn^2 + O(1), \quad n \to \infty,$$

for some $c > 0$ (see [16, Remark 1] and [32]). Clearly, the addition of a point to the spectrum amounts to shifting the enumeration by 1, producing an asymptotic formula with an additional nontrivial term linear in n not present in (37). An analogous reasoning holds for the case of Dirichlet boundary condition at b. We remark that Theorem 6.2 holds for a wide class of perturbations of the regular Laplacian for which we do not have at our disposal an asymptotic formula of the kind of (37) for the spectra of the corresponding selfadjoint extensions. The asymptotics for Schrödinger operators with singular potentials, which we are aware of (see [38, Theoreom 4]), are not sufficiently precise for using the argumentation given above to exclude the possibility of adding an eigenvalue.

The example presented here is also related via [34, §17] to the Gelfand–Levitan theory for half-line Schrödinger operators. Let us discuss this in detail. In [12] and [34, Sec. 17] the spectral measure of the half-line Schrödinger operator is defined as follows. Suppose in (36) that $q \in L_{1,\text{loc}}([0, \infty))$ (with fixed boundary conditions at 0) and take into account that $\xi(x, z)$ is the same for any right endpoint $b \in \mathbb{R}$. A spectral measure of the half-line Schrödinger operator is a measure ρ such that the Parseval identity

$$\|\varphi\|^2_{L_2(0,b)} = \|f\|^2_{\mathcal{B}_A} = \int_{\mathbb{R}} |f(\lambda)|^2 \, d\rho(\lambda) \tag{38}$$

holds for every $f = \Phi_A \varphi \in \bigcup_{b>0} \mathcal{B}_A$ (see (33) and compare with [12, Equation 2] and [34, Theorem 3.2]). The classical Weyl theory tells us that a Schrödinger expression has exactly one spectral measure if q is in the limit point case at infinity, otherwise it has an infinite set of spectral measures. We remark that in the limit circle case, the definition of spectral measure given above includes measures which do not correspond to selfadjoint operators in $L_2(0, \infty)$ (see in [12, §8] complementary conditions for a spectral measure to correspond to selfadjoint extensions).

The central result in Gelfand–Levitan theory is a theorem on necessary and sufficient conditions for a measure to be a spectral measure of a half-line Schrödinger operator, that is, the necessary and sufficient conditions on ρ such that there is a potential function $q \in L_{1,\text{loc}}([0, \infty))$ ensuring that (38) holds. Due to the nonlocality of the conditions (see [12, Introduction]), if ρ is a spectral measure of a half-line Schrödinger operator, then there is a potential such that $\rho + s\delta_\lambda$ (with $\lambda \notin \text{supp} \, \rho$ and $s > 0$) is the spectral measure of the half-line Schrödinger operator with that potential and same boundary condition at 0 (see [12, §7, Example 2] and [26, Chapter 2, §7]).

Note that the last statement is not in contradiction with Theorem 6.2. Actually, the results of this paper and the theory of perturbations of chains of de Branges spaces developed in [48] can be used to study the point mass perturbations of spectral measures of a half-line Schrödinger operator. This remains to be done.

To conclude this example, let us revisit the assertion of Theorem 6.2(a) at the light of [4, §5, 6]. It is now relevant to keep track of the signed measure defining the operators, so let us denote by $A(\mu)$ the operator given in (32). By [4, Theorems 5.2, 6.1 and 6.2], we know that there is a bijective correspondence between the set of signed Borel measures μ in $[0, b]$, and the set of even functions v on $(-2b, 2b)$ of bounded variation such that

$$v(0) = -\mu(\{0\}) \tag{39}$$

and the operator $I + K_v$ is positive definite, where

$$(K_v \varphi)(x) = \frac{1}{2} \int_0^b (v(t - x) + v(t + x)) \varphi(t) \, dt.$$

If μ and v are connected by this correspondence, then

$$\|f\|^2_{\mathcal{B}_{A(\mu)}} = \langle \Phi_{A(0)}^{-1} f, (I + K_v) \Phi_{A(0)}^{-1} f \rangle_{L_2(0,b)}.$$

Theorem 6.4. *Assume the same hypothesis of Theorem 6.2. Then, there is a signed Borel measure v such that $\tilde\rho$ is a nonextremal solution to the generalized moment problem for $\mathcal{B}_{A(v)}$.*

Proof. In view of Theorem 6.2(a), it only remains to prove the existence of a signed Borel measure ν such that, for the corresponding space $\mathcal{B}_{A(\nu)}$, the equality

$$\|f\|^2_{\mathcal{B}_{A(\nu)}} = \int_{\mathbb{R}} |f(\lambda)|^2 \, d\tilde{\rho}(\lambda) \tag{40}$$

holds. From the linearity of K_ν as a map on the function ν, we can assume without loss of generality that ρ_γ is given by (34) for a selfadjoint extension of $A(0)$. Thus,

$$\|f\|^2_{\mathcal{B}_{A(\nu)}} = \|\Phi^{-1}_{A(0)} f\|^2_{L_2(0,b)} + s |f(\lambda)|^2.$$

According to [4, Theorem 6.1] (see above), the proof will be established, once we find an even function ν on $(-2b, 2b)$ of bounded variation such that

$$\langle \Phi^{-1}_{A(0)} f, K_\nu \Phi^{-1}_{A(0)} f \rangle = s |f(\lambda)|^2.$$

However, using the notation introduced in Section 4,

$$s|f(\lambda)|^2 = s \langle \Phi^{-1}_{A(0)} f, \xi_{A(0)}(\lambda) \rangle \langle \xi_{A(0)}(\lambda), \Phi^{-1}_{A(0)} f \rangle = \langle \Phi^{-1}_{A(0)} f, s \langle \xi_{A(0)}(\lambda), \Phi^{-1}_{A(0)} f \rangle \xi_{A(0)}(\lambda) \rangle.$$

So K_ν must be equal to $s \langle \xi_{A(0)}(\lambda), \cdot \rangle \xi_{A(0)}(\lambda)$. Therefore, using the fact that $\xi_{A(0)}(\lambda, \cdot) = \cos(\sqrt{\lambda}\,\cdot)$, we have

$$\begin{aligned}
(K_\nu \varphi)(x) &= s \int_0^b \xi_{A(0)}(\lambda, t) \xi_{A(0)}(\lambda, x) \varphi(t) \, dt \\
&= s \int_0^b \cos(\sqrt{\lambda} t) \cos(\sqrt{\lambda} x) \varphi(t) \, dt \\
&= \tfrac{1}{2} s \int_0^b (\cos(\sqrt{\lambda}(t+s)) + \cos(\sqrt{\lambda}(t-s))) \varphi(t) \, dt,
\end{aligned}$$

which yields $\nu(\cdot) = s \cos(\sqrt{\lambda}\,\cdot)$. $\qquad\square$

For the function ν found in the proof above, $\nu(0) \neq 0$. Thus, according to Remark 6.3 and (39), the point mass perturbation of an extremal spectral measure of a regular Schrödinger operator is a nonextremal spectral measure of a nonregular Schrödinger operator with measure-valued potential.

6.3. Bessel operators. Given $b \in (0, \infty)$, consider the differential expression

$$\tau_q := -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2} + q(x), \quad x \in (0, b), \quad \nu \in [0, \infty). \tag{41}$$

We assume that $q \in L_{1,\mathrm{loc}}(0, b)$ is a real-valued function such that $\tilde{q} \in L_1(0, b)$, where

$$\tilde{q}(x) := \begin{cases} xq(x) & \text{if } \nu > 0, \\ x(1 - \log(x))q(x) & \text{if } \nu = 0. \end{cases} \tag{42}$$

As shown in [18, Theorem 2.4], τ_q is regular at $x = b$ whereas at $x = 0$ it is in the limit point case if $\nu \geq 1$ or in the limit circle case if $\nu \in [0, 1)$. For the later case we assume the additional boundary condition

$$\lim_{x \to 0+} x^{\nu - 1/2}\big((\nu + \tfrac{1}{2})\varphi(x) - x\varphi'(x)\big) = 0. \tag{43}$$

The expression (41), along with the boundary condition (43) when $v \in [0, 1)$, gives rise to a closed, regular, symmetric operator A, whose deficiency indices $(1, 1)$ [41, §4].

The corresponding canonical selfadjoint extensions A_γ are defined as usual,

$$(44) \qquad \mathrm{dom}(A_\gamma) := \left\{ \begin{array}{c} \varphi \in L^2(0, b) : \varphi, \varphi' \in \mathrm{AC}[0, b], \ \tau_q\varphi \in L^2(0, b), \\ \text{boundary condition (43) if } v \in [0, 1), \\ \varphi(b)\cos\gamma + \varphi'(b)\sin\gamma = 0, \ \text{fixed } \gamma \in [0, \pi) \end{array} \right\}, \quad A_\gamma\varphi := \tau_q\varphi.$$

For every $\gamma \in [0, \pi)$, the spectrum of A_γ, beside of being simple and discrete, has at most a finite number of negative eigenvalues [18, Theorem 2.4].

By [18, Lemma 2.2], the eigenvalue equation $\tau\varphi = z\varphi$ $(z \in \mathbb{C})$ admits a solution $\xi(z, x)$, real entire with respect to z, with derivative $\xi'(z, x)$ also real entire. Moreover, $\xi(z, x)$ also obeys boundary condition (43) whenever $v \in (0, 1)$. This in turn implies $\xi(z, \cdot) \in \ker(A^* - zI)$ for all $z \in \mathbb{C}$. Hence, the associated de Branges space is

$$(45) \qquad \mathcal{B}_A := \left\{ f(z) = \int_0^b \xi(z, x)\varphi(x)dx : \varphi \in L_2(0, s) \right\}, \quad \|F\|_{\mathcal{B}_A} = \|\varphi\|_{L_2(0,b)}.$$

Also, the extremal solutions of the corresponding generalized moment problem are given by

$$(46) \qquad \rho_\gamma = \sum_{\lambda \in \mathrm{spec}(A_\gamma)} \frac{\delta_\lambda}{\|\xi(\cdot, \lambda)\|^2_{L_2(0,b)}},$$

where A_γ is any selfadjoint extension of A. We now have the following statement analogous to Theorem 6.2.

Theorem 6.5. *Suppose $v > 0$, and $q \in L_{1,loc}(0, b)$ such that $\tilde{q} \in L_r(0, b)$ for some $r \in (2, \infty]$. Let A be the Bessel operator defined by τ_q as in (32). Let A_γ be an arbitrarily chosen selfadjoint extension of A and ρ_γ the corresponding spectral measure given in (46). If*

$$\tilde{\rho} = \rho_\gamma + s\delta_\lambda$$

with $\lambda \notin \mathrm{spec}(A_\gamma)$ and $s > 0$, then:

(a) *There is no $p \in L_{1,loc}(0, b)$ with $\tilde{p} \in L_r(0, b)$ for some $r \in (2, \infty]$ such that $\tilde{\rho}$ is the spectral measure corresponding to selfadjoint extensions of $\tilde{A}$ generated by τ_p.*

(b) *The set of entire functions*

$$(47) \qquad \left\{ f(z) := \sqrt{\tfrac{\pi}{2}} z^{-v/2} \int_0^b \sqrt{x} J_v(\sqrt{z}x)\varphi(x)\, dx : \varphi \in L_2(0, b) \right\}$$

is a proper subset of $L_2(\mathbb{R}, \tilde{\rho})$, where J_v denotes the Bessel function of the first kind.

Proof. The assertions follow from Theorem 4.12, Theorem 5.3 and Theorem 5.4, this time combined with [41, Theorem 4.2]. $\qquad \square$

Remark 6.6. The requirement on $\tilde{q}$ stated in Theorem 6.5 is a technical limitation related to the perturbative argument used in the proof of [41, Theorem 4.2]. The technique used there can be easily modified to include $r = 2$ but breaks down for $r \in [1, 2)$.

We emphasize that our theory does not rule out the possibility that $\tilde{\rho}$ is the spectral measure of a Bessel operator for some different value of the parameter ν. In this respect, it is worth noticing the following: Assuming $r \geq 2$ and resorting to [18, Theorem 2.5], one can show that the eigenvalues of A_γ obey the asymptotic formula

$$(48) \qquad \lambda_n = \frac{\pi^2}{b^2}(n + \kappa_\nu)^2 + O(n^{1/2}), \quad n \to \infty, \kappa_\nu := \begin{cases} (2\nu + 1)/4, & \gamma \neq 0, \\ (2\nu - 1)/4, & \gamma = 0. \end{cases}$$

This implies that, contrary to the case of regular Schrödinger operators, the addition of a point to the spectrum is still compatible with (48) but for a different value of ν, namely, for $\tilde{\nu} = \nu + 2$.

Acknowledgments

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RAFAEL DEL RIO: delrio@iimas.unam.mx
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Departamento de Física Matemática,
Universidad Nacional Autónoma de México, Mexico City, Mexico

LUIS O. SILVA: silva@iimas.unam.mx
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Departamento de Física Matemática,
Universidad Nacional Autónoma de México, Mexico City, Mexico

JULIO H. TOLOZA: julio.toloza@uns.edu.ar
Instituto de Matematica (INMABB), Departamento de Matematica, Universidad Nacional del Sur (UNS) — CONICET,
Bahia Blanca, Argentina